Under what conditions does the bar model support mathematical problem solving of two-step, real-life, word problems for autistic students?

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Abstract

The current PhD thesis explores the key conditions (factors) associated with mathematical problem solving amongst autistic pupils, with a focus on the use and application of the bar model. Previous research within mathematical problem solving identifies an uneven range of profiles amongst the autistic population. Furthermore, a lack of empirical evidence exists as to the success and best classroom practice within mathematical problem solving for this group of pupils. Previous studies identify individual conditions likely to be influential on mathematical problem solving, the aim of the current study was to explore the combinations of these conditions with respect to successful word problem solving amongst autistic pupils. Although the bar model is becoming more widely adopted within mathematics teaching and learning, there remains a paucity of empirical research around the conditions associated with its success. The research questions are answered through the exploratory use of qualitative comparative analysis (QCA) in small-N (N=9), educational research. Through the use of pupil discussions and observations of mathematical problem solving and teacher interviews, the analysis of individual conditions and configurations of conditions giving rise to successful mathematical word problem solving are explored. The findings from the study identify mathematical attainment and pupils' self-perception of their own mathematical ability to be significant in mathematical problem solving. Through further analysis, the impact of the executive functions, particularly working memory and attention, are identified, along with the importance of pupils' conceptual understanding within mathematical problem solving. The study suggests that teachers pay particular attention to the broader profiles of autistic pupils within the mathematics classroom and consider carefully the balance of procedural and conceptual teaching and learning, particularly when utilising the bar model. Through a range of data analysis techniques, the study provides encouraging data to advocate the use of QCA in small-N, educational research.

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List of abbreviations

AS	Asperger’s syndrome
ASD	Autism spectrum disorder
BM	Bar model
CC	Central coherence
CSI	Cognitive strategy instruction
csQCA	Crisp-set Qualitative Comparative Analysis
DSM-5	Diagnostic and statistical manual of mental disorders (version 5)
EF	Executive functions
fsQCA	Fuzzy-set Qualitative Comparative Analysis
HFA	High functioning autism
ICD-10	International classification of diseases (version 10)
MA	Mathematical attainment
NT	Neurotypical
PISA	Programme for international student assessment
QCA	Qualitative Comparative Analysis
RA	Reading attainment
SBI	Schema-based instruction
TD	Typically developing
TIMSS	Trends in international mathematics and science study
ToM	Theory of mind
VROF	Visual representation observation form
WCC	Weak central coherence

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Figure 46: Refinement of the conceptual framework, based on the findings from the current study (p.287)

Chapter 1: Introduction

The current study sets out to explore the key contextual factors and mechanisms, or tendencies, underpinning successful mathematical word problem solving for autistic pupils. As a thread running throughout the study, a focus on the use of the bar model, as a visual representation in mathematics is explored. The study aims to address some of the gaps in the current literature and empirical studies, pertaining to these areas, which are identified below.

The study focus was chosen based on my personal experiences of supporting autistic pupils in the mainstream classroom and my teaching of mathematics to trainee teachers in higher education. As a former headteacher of a small primary school, one of the key challenges I faced was the successful inclusion of a growing number of pupils diagnosed with autism, along with the drive to change the mindset of class teachers to focus on these pupils' individual strengths, rather than their deficits. The ongoing pressure of end of year data sets, yet the need for successful inclusion and parity in pupils' experiences, provided a constant challenge within the primary school.

Coupled with these challenges as a headteacher, my own teaching of mathematics to trainee primary teachers has enabled me to explore best practice in mathematics' teaching and learning, including the use of the bar model, as a visual representation in mathematics. These two personal experiences have become the driver for the current PhD study and for further personal research beyond, to explore ways in which the successful inclusion and teaching and learning of mathematics for autistic pupils can be maximised within the primary school years.

The introduction begins by providing the reader with a background and contextual overview of the current position regarding the education of autistic pupils within mainstream settings; the current landscape of mathematics education in English primary schools; and an overview of the development of the bar model, and its use as a tool for visually representing mathematical concepts. Following on from this, a clear justification for the study is provided, identifying key gaps in the current literature, which this study seeks to address. The analytical framework is then discussed, identifying a further contribution to knowledge in terms of the methodological approach used within the current study. Finally, an overview of the structure of the thesis is presented to provide a summary of the remainder of the study for the reader.

1.1 Background of the study

1.1.1 Autism in English mainstream primary schools

There has been a notable rise in the number of students being diagnosed with an autism spectrum disorder (ASD) (Lindsay et al., 2013) with a steady increase over the last four decades (Baron-Cohen et al., 2009); more than 70% of these pupils attending mainstream schools (APPGA, The National Autistic Society, 2017, p. 9). Recent government data suggests that 14.4% of pupils educated in English, state funded, mainstream primary schools, have a recognised Special Educational Need (SEN), with an average of 4.64% of all pupils in these settings having a diagnosis of autism (Department for Education, 2017).

Furthermore, those autistic individuals are held accountable to the same academic standards comparable to their peers (Schaefer-Whitby, 2013). This increase in awareness and diagnosis has driven the need for a more 'inclusive education', which Lindsay et al. (2013) suggest, for pupils diagnosed with ASD, can lead to 'increased student engagement in social interaction, higher levels of social support, social networks and advanced education goals compared with their counterparts in segregated settings' (pp. 347–348). In a recent review of current priorities within autism research, educational support was ranked 5th, therefore suggesting the need for further research within this area (Wallace et al., 2019). Lindsay et al. (2013) argue that applying best-practice elements of inclusion, which incorporates curricular adaptation, may be particularly difficult where students with higher cognitive abilities are presented, as many of the resources provided to drive an inclusive curriculum are guided towards those individuals with lower cognitive abilities. Despite an increase in the number of studies focusing on educational interventions for autistic pupils, there is a lack of research carried out within the general education and classroom contexts (Gevarter et al., 2016; Wallace et al., 2019). Consequently, the present study focuses directly on classroom practice and performance within mathematics for this group of pupils.

1.1.2 The current landscape of mathematics education in English primary schools

With increasing pressure on schools to ensure that students gain high levels of attainment by the end of year six (age 11), this study aims to explore some of the necessary[1] and sufficient[2] conditions[3] required, with a focus on the bar model as one type of visual representation, for autistic pupils to solve real life, mathematical word problems to enable them to make expected or better progress in mathematics.

According to the DfE (2016), whilst there is no 'target' for the amount of progress an individual pupil is expected to make in mathematics (or any other subject) each year, the current progress measures aim to capture the progress that pupils make from the end of key stage 1 to the end of primary school. These measures of progress are a 'type of value added measure, which means that pupils' results are compared to the actual achievements of other pupils nationally with similar prior attainment.' (Department for Education, 2016, p. 4). Furthermore, those 'pupils with special educational needs (SEN) tend to demonstrate a larger attainment gap when compared to those without any identified SEN' (p.26).

[1] Although these terms are discussed later, as support for the reader, they are simplified here: a 'necessary' condition is 'always present when the outcome occurs (i.e. the outcome cannot occur in the absence of the condition)' (Rihoux & Ragin, 2009, p. xix).

[2] A 'sufficient' condition 'always occurs when the outcome is present. However, the outcome could also result from other conditions.'

[3] The term 'conditions' is synonymous with factors. As the current study is based on the methodological framework of Qualitative Comparative Analysis (QCA), the terminology of 'condition' is applied. A condition refers to a factor, which gives rise to the a desired, and specified, outcome.

Following the publication of The Cockcroft Report (Cockcroft, 1986), a recommendation for a greater emphasis on problem solving in primary schools has given rise to significant curriculum reform and rethinking, resulting in an updated curriculum, with a strong focus on mathematical problem solving (Department for Education, 2016). Much debate continues to exist as to what problem solving actually is (Gerofsky, 1996a; Gningue et al., 2014a; F. K. Lester & Cai, 2016; A. Schoenfeld, 2018; Shaughnessy, 1985a). It appears common place in primary schools that problem solving is often interpreted quite simply as finding the solution to a word-problem. However, this understanding and viewpoint has been challenged (Gerofsky, 1996; Gningue et al., 2014; Lester & Cai, 2016; Shaughnessy, 1985) and has led to a better understanding of what problem solving actually is (discussed in chapter 2.2).

Coupled with the implementation of the new curriculum (DfE, 2013), a shift of focus in the teaching and learning of mathematics has driven given rise to a greater emphasis on problem solving and problem-posing, which Rosli et al*.* (2013) consider to be important cognitive activities within mathematics teaching and learning. Traditional mathematics teaching tended to consider problem solving as a discrete area of mathematics, often used as an assessment tool and more commonly with older, or more able, pupils. Furthermore, in the 2000's, the focus of problem solving was often on content, rather than mathematical processes and reasoning (Pratt & Woods, 2007). More recently however, it is suggested that the application of four processes are required for successful problem solving: reasoning, communication, connections and representations (Mutawah et al., 2019, p. 258), aligning closely with current guidance on teaching for mastery and embedding problem solving into the mathematics curriculum (NCETM, 2019). In their mixed methods study (N=350), Mutawah et al*.* (2019), concluded that through using multiple representations to introduce mathematical concepts, students demonstrated a deeper understanding of such concepts, supporting Barmby et al.'s (2013) earlier work, suggesting that the ability to draw on multiple representations constitutes a significant factor in children's mathematical understanding. In line with curriculum development, assessment of pupils' mathematical abilities has begun to shift from a traditional emphasis on mathematical skills and procedures to a greater focus on problem solving (NCETM, 2019; Pratt & Woods, 2007; Rosli et al., 2013). Furthermore, in Schoenfeld's (2018) later work, when considering Polya's (1945) problem solving process, he argues that the ability to 'monitor and assess' one's actions whilst engaging in problem-solving steps, plays a significant role in determining overall success (Schoenfeld, 2018, p. 290). Lester (1985) proposes that such 'metacognitive decisions serve to guide cognitive actions' (p.64), thus aligning very closely with the theory of executive function (EF) (discussed in chapter 2.1.2), in terms of its potential impact upon some autistic individual's abilities in problem solving (Berenguer et al., 2017; Gioia et al., 2000; Goldstein & Naglieri, 2014). Lester (1985) advocates that the ability to 'monitor and evaluate the progress and outcome of the tactics and plan' are a requirement for 'competent' problem-solvers (p.43). Nevertheless, despite much problem solving instruction having its foundations in Polya's (1945) work, there is an argument that suggests this approach is too general and does not provide the support or metacognitive skills needed for students with learning disabilities to become successful problem solvers (Fuchs et al., 2006; Jitendra, 2008; Spooner et al., 2017).

When it comes to solving mathematical word problems, the formation of, and access to information stored within schemas appears to play a significant role in overall success (discussed in detail in chapters 2.2.2 and 2.2.2) (Marshall, 1995; Mayer, 1985; Peltier & Vannest, 2018; Schoenfeld & Herrman, 1982). Coupled with schema development and retrieval, is the role of cognitive load in terms of storing information from the problem within the short-term buffer, whilst processing information assigned within schemas (Ngeno et al., 2019). It is suggested that a reduction in cognitive load may be supported with visual representations by utilising both visual and verbal pathways. Furthermore, the use of external schematic representations have been positively correlated with problem-solving success (Hegarty & Kozhevnikov, 1999; Peltier & Vannest, 2018; Siregar et al., 2019). It is suggested that the use of visual or schematic representations, of which the bar model is one example, can assist with the development of new schemas and comprehension of word problems (Kintsch & Greeno, 1985; Maglicco, 2016; Peltier & Vannest, 2018; Siregar et al., 2019).

It is clear from the literature that different sub-groups of the population demonstrate different problem-solving abilities and is further compounded by the effectiveness that different types of visual representation may offer (Cooper et al., 2018a). The ability to make an appropriate choice of strategy, or representation, may indeed be a necessary factor for problem-solving success (Schoenfeld & Herrman, 1982).

When considering the reform and development seen in the English National Curriculum for mathematics (DfE, 2013), significant influence is driven by the results of international comparative assessments of academic performance such as the Programme for International Student Assessment (PISA) (DfE, 2016). Influences from those countries who demonstrate higher levels of performance than England, particularly in mathematics, of which Singapore is one (mean mathematics scores: Singapore: 564; England: 493 (DfE, 2016, p. 66)), frequently impact upon the classroom practice and curriculum development of schools in England.
Rising from one of the lowest performing countries in terms of mathematics, Singapore's curriculum development and focus on mathematical achievement, led to them being ranked first in the 1995 TIMSS assessment, where they have maintained their high performance ever since, following the introduction of the Singapore mathematics curriculum framework (SMCF) in 1990 (Fan & Zhu, 2007). Similar to in England (DfE, 2013), performance in international comparative tests, such as TIMSS (Trends in International Mathematics and Science Study) and PISA (Programme for International Student Assessment), have been the primary driver of curriculum reform in Singapore (OECD, 2012; Singapore Ministry of Education, 2012). The resulting reforms have given rise to a greater emphasis on the development of pupils' deep understanding of mathematical concepts and applications within the Singapore mathematics curriculum:

'It is the goal of the national mathematics curriculum to ensure that all students will achieve a level of mastery of mathematics that will serve them well in their lives, and for those who have the interest and ability, to pursue mathematics at the highest possible level.' (Singapore Ministry of Education, 2012, p. 2).

When comparing the mathematics curricula for England and Singapore, it could be argued that both contain a large amount of content, however, when analysing the linguistics within these two curricula, some differences appear to emerge (DfE, 2013; Singapore Ministry of Education, 2012). A focus in the Singapore curriculum is heavily upon understanding ('Pupils should be *taught to understand*', rather than 'Pupils should be *taught*', as stated in the English curriculum). Furthermore, within the Singapore curriculum, there is a focus on the terminology of 'notation', 'representations', 'relationships' and 'determining'- terms which emphasise a secure understanding of concepts and the ability to develop relationships between concepts and application to a range of contexts. When the linguistic information within the English curriculum is analysed more closely, terms such as 'read', 'write', 'order', 'count' and 'know' are frequently used - terms which do not necessarily suggest depth of understanding, but in some cases, merely recall of facts.

However, the potential context-specific nature of the PISA results and therefore the external validity and interpretation of the findings must be acknowledged. In addition to Singapore and other East Asian countries, other countries, such as Switzerland and Estonia (with mean scores of 523 and 520 respectively) are also high performing countries in mathematics (DfE, 2016, p. 65). However, little attention is paid to these countries in terms of influencing the English mathematics curriculum.

One visual representation in mathematics, developed and commonly used within mathematics' teaching in Singapore, is the bar model, which 'consists of a series of rectangles in which the relationships of the rectangles are specified and presented globally' (Ng & Lee, 2009, p. 285), enabling the global context of the problem to be encompassed. Although a recent study was carried out in an attempt to identify the reasons behind such performance gaps between England and East Asian jurisdictions (Jerrim & Choi, 2014), there is no evidence, at present, to suggest that the mathematical performance in Singapore schools is solely, if at all, a consequence of the bar model approach.

However, the emphasis on mathematical reasoning and problem solving in the current National Curriculum guidance (DfE, 2013) has given rise to an increased number of schools adopting the bar model, sometimes referred to as 'The Singapore Bar Model', as an approach to supporting mathematical understanding and problem solving. The current study seeks to explore the likely influence of such mathematical representation within the overall problem-solving ability, with a focus on autistic pupils.

1.2 Justification for the study

Wallace et al. (2019) suggest that many of the interventions, on which much current research is based, are not designed within the classroom or school environment, and are often not implemented in the way in which they were designed. Due to this mis-alignment between intervention research design and implementation in the school context, there is a subsequent lack of empirical evidence to drive policy and practice, with respect to 'what best practice means in real-world classrooms' (Wallace et al., 2019, p. 50). Consequently, in line with the recommendations made by Wallace et al. (2019), the current study considers the potential implications of the bar model, as a mathematical visual representation, within the classroom context, for which it is designed. The challenges of such inclusive practice are also supported by Agrawal (2013), where recognition is paid to the heterogeneity of cognitive abilities of those pupils on the autistic spectrum. Despite their research being carried out in a Canadian context, the search for inclusive curricula within the English education system aligns closely with these findings, as many autistic pupils are likely to be educated within the mainstream classroom (APPGA, The National Autistic Society, 2017; Ozier, 2013).

One of the main challenges facing researchers in the area of autism, is governed by the heterogeneity of the condition itself, coupled with the uneven profiles of abilities across different domains, including mathematics (Aagten-Murphy et al., 2013; Agrawal, 2013; Chiang & Lin, 2007; Wallace et al., 2019; Wen, 2018; Whitby & Mancil, 2009). Furthermore, as yet, any specific factors, or sub-groups, directly linked with mathematical ability amongst the autistic population, remain unclear (Keen et al., 2015; Powell et al., 2019; Wei et al., 2015), which the current study aims to address.

A search of the literature has revealed that, as yet, the use of the bar model, as a tool for supporting problem solving abilities within the autistic population, has not been fully explored. Consideration is given to underlying cognitive theories of autism (discussed in detail in Chapter 2.1.2), such as Theory of Mind (ToM), Executive Functioning (EF) and Weak Central Coherence (WCC) (Levy, 2007), in line with various proposed models of solving mathematical word problems (Kintsch & Greeno, 1985; Polya, 1945; Skemp, 1978) (discussed in Chapter 2.2). This study begins to explore whether a representational model, such as the bar model, may offer support for autistic pupils in terms of deconstructing the problem into detailed, focused parts, hence increasing their problem-solving success.

When it comes to solving mathematical word problems, Jitendra et al. (2007) point out that there is a requirement for the integration of several cognitive processes, including reliance on reading comprehension, understanding abstract concepts, higher order thinking and written expression, mathematical vocabulary, and everyday mathematical knowledge, all of which, according to the authors, are known to be potential areas of weakness for autistic pupils. However, phonological decoding has not been identified as an area of weakness amongst this population (Agrawal, 2013; Bae, 2013; Chiang & Lin, 2007). In contrast to the effectiveness of short-term intervention, little research, particularly which encompasses qualitative elements, has been carried out into the field of more general teaching approaches and strategies to support the mathematical understanding of autistic pupils. Research to examine the specific factors, which may be predictors of academic success within this population, remains limited (Keen et al., 2015). Much of the existing research literature relating to autistic pupils and mathematics, relates directly to the acquisition of specific mathematical skills or the impact of specific short-term interventions, rather than the most effective and efficient strategies of teaching mathematics generally, or the factors associated with positive outcomes for autistic individuals (Aagten-Murphy et al., 2013; Agrawal, 2013; Bae, 2013; Tzanakaki et al., 2014; Whitby & Mancil, 2009).

Given this increase in use of the bar model within mathematics teaching and learning (NCETM, 2021), combined with the rising number of autistic pupils within mainstream classrooms, this study seeks to address some of the gaps in current research and understanding into the specific-factors, which impact mathematical problem-solving for this population, whilst analysing the current trends in classroom practice, in an attempt to bridge the gap between research and practice for autistic pupils.

A focus on the bar model, as a tool for visualising mathematical problems, may build upon the findings from previous research into difficulties faced by pupils with autism with mapping numerical representations into space and difficulties in visuo-spatial understanding (Aagten-Murphy et al., 2015; Agrawal, 2013). Furthermore, the current study aims to support the need for further research into potential factors to support mathematical word problem solving ability for autistic students (Bae et al., 2015; Whitby & Mancil, 2009) as a prerequisite for supporting these pupils in the general mathematics curriculum.

Consequently, whilst significant research has been previously carried out to determine the overall problem solving and mathematical ability of pupils with autism and the effectiveness of short-term interventions (Aagten-Murphy et al., 2013; Agrawal, 2013; Bae et al., 2015), recent evidence suggests that there is a distinct lack of research to 'bridge the gap between understanding the nature of academic achievement for individuals with ASD and working with educators to create practices that support autistic individuals to achieve academic success' (Keen et al., 2015, p. 17).

1.3 Analytical framework

The current study is positioned within a critical realist perspective, which seeks to uncover the potential causal mechanisms, or tendencies, which give rise to empirical observations (Anderson, 2019). As the study seeks to explore the conditions, or combinations of conditions, influencing mathematical problem solving amongst autistic pupils, the research is grounded within the methodological framework of qualitative comparative analysis (QCA), in which its theoretical foundations align firmly with the critical realist framework of causation (Jopke & Gerrits, 2019). The use of QCA, as a methodological framework within small-N educational research, is a relatively unexplored area, hence as a partial contribution to knowledge within the current thesis, the study also provides a critique of this research approach. Aligning with critical realism, QCA, as a methodological approach, sets out to explore the multiple complex configurations of conditions (which may be contextual or mechanistic) necessary to give rise (or not) to a specific outcome (or action) (in the case of the current study – successful mathematical problem solving) (Anderson, 2019). Therefore, what CR seeks, and what QCA offers in the current study, is an understanding of the causes of an outcome (or action) when the conditions are right, and the generative mechanisms underlying the observable outcome (Anderson, 2019; Harré & Madden, 1975; MacLeavy, 2019).

1.4 Research questions

Based on the current gaps in knowledge and understanding, discussed above, the current study aims to answer the following research questions:

1. **What are the key contextual factors and mechanisms underpinning successful solving of mathematical word problems for autistic pupils?**

2. **Can the exploratory use of qualitative comparative analysis (QCA) be used to determine sufficient and necessary conditions required for autistic pupils solve mathematical word problems?**
   a. Is the bar model sufficient, or does it form a necessary factor within a combination of other conditions, for autistic pupils to solve two-step, real-life mathematical word problems?
   b. Is the bar model sufficient to support autistic pupils in solving mathematical word problems?
   c. Does the bar model form a necessary factor within a combination of other conditions to support autistic pupils in solving mathematical word problems?

3. **Is the overall success rate in determining the correct solution to mathematical word problems greater when the bar model is employed by autistic pupils?**
    a. Do autistic pupils, who have been exposed to the bar model, choose this approach when solving mathematical word problems?
    b. Do autistic pupils choose the use of visual representations when solving mathematical word problems?

1.5 Outline of the thesis

The thesis begins with a comprehensive review of the literature (Chapter 2), providing the current understanding of the three main areas in which this study is grounded: autism; mathematical problem solving; and the bar model, as a visual representation.

After establishing the terminology to be used within the current study, chapter 2.1 establishes an understanding of autism from both a medical- and social-perspective, before moving on to provide an overview of the mathematical profiles often seen amongst this population.

Having discussed some of the key relevant theories underpinning problem solving and considering some of the pedagogical implications deemed to be significant within the classroom, chapter 2.2 goes on to explore mathematical word problems, as one type of problem solving commonly associated with the application of the bar model. Using specific, worked examples, along with analysis of some previous studies, the complexities of problem-solving through the use of mathematical word problems begins to emerge. Some key theories and processes underpinning word problem-solving are discussed, with clear links to some of the theories associated with problem solving generally. A discussion of word problems as a unique genre of textual information, combined with the influence of the classroom environment and implicit rules involved within their understanding, are explored. Together, this enables the complexities associated with word problem-solving in mathematics, as well as an understanding and explanation of some of the difficulties faced by pupils within this domain, to be drawn upon to provide a clear conceptual framework, on which the current study is based.

Chapter 3, after introducing the philosophical perspective of critical realism, discusses, in detail, QCA as a methodological framework. The theoretical foundations of QCA are discussed in-depth, providing a clear rationale for its use within the current study. Following on from this, the data collection used within the current study is discussed. Chapter 3.5 provides the reader with a detailed overview of the preliminary study, which was used to refine the research design and to establish the measures to be used within the data analysis. Chapter 3.6 explores data analysis, both in terms of coding and the use of QCA, in detail. Finally, this chapter provides an overview of the ethical considerations within the current study and a clear position on validity, rigour and trustworthiness for the reader.

The findings from the current study are discussed, at length, in chapter 4. Within this chapter, the findings are presented in line with the data collection and analysis approaches applied within the study.

Chapter 5 positions the findings from the current study with the current body of literature, to support the refinement of the conceptual framework introduced in chapter 2.3. Finally, a critique of QCA, as a methodological approach in small-N educational research is provided.

Chapter 6 draws together the study, proving a clear conclusion of the findings, thus answering the research questions above. The limitations of the study are acknowledged before discussing the implications for classroom practice and identifying the potential areas for future research emerging from the study.

Chapter 2: Literature Review

The literature reviewed is structured around the three broad themes within this study: autism; mathematical problem solving; and the bar model, as one type of visual representation within mathematics. Each of these areas is discussed in turn, drawing on previous research and empirical studies to provide a comprehensive overview of the current understanding and gaps in knowledge in each area.

I begin by reviewing the existing literature around autism as a spectrum disorder, to establish a current understanding of the condition. One of the challenges faced by many researchers within this field, is the debate over terminology associated with autism. This chapter therefore goes on to provide an overview of some of the recent arguments within this field, before moving on to consider autism from a medical model perspective, which discusses some of the cognitive theories associated with autism. Following on from this, autism is considered from a social model perspective, to provide an alternative explanation for the way in which the condition is presented. By drawing on both medical (cognitive) and social theories, a clear definition of autism, for the purposes of this study, is reached.

Next, a review of the literature around the mathematical profiles of autistic individuals is carried out, to establish the potential educational implications in the mainstream classroom associated with the condition. Furthermore, the findings from this part of the literature review are used to guide the design of the research tools for data collection.

Finally, the chapter then goes on to consider mathematical problem solving and visual representations within mathematics, before exploring the considerations of the bar model as a visual representation within mathematical problem solving. Through drawing on these three broad areas, a conceptual framework, on which the current study is based, is developed. The development of the conceptual framework is used to guide the subsequent research design and process.

2.1 Autism Spectrum Disorder (ASD): Current theory and debates

Autism spectrum disorder (ASD) is an 'early onset neurodevelopmental condition, characterised by altered social communication and interaction, alongside restricted and repetitive behaviours and interests, causing significant functional impairment' (Bölte et al., 2018, p. 1). A significant proportion of traditional research into autism has tended to focus on deficits, prevention or cures and research frequently considers the autistic population as a homogeneous group (Wallace et al., 2019; Wen, 2018). According to surveys carried out by the Autism Education Trust (AET) involving young autistic people and adults, carers and practitioners (N=900), autistic behaviours are frequently 'underpinned by difficulties in both the flexible generation of ideas and the understanding of, and thinking about, other people's thoughts and feelings' (Wittemeyer et al., 2011, p. 11). Furthermore, the report also acknowledges the frequent occurrence of 'lower or heightened sensitivity to sensory information, and interests in particular sensations' (p.11).

In order to understand how these characteristics may present themselves, table 1 exemplifies some of the criterial features, which form part of the assessment guidance for autism within the Diagnostic and Statistical Manual of Mental Disorders (DSM-5) (American Psychiatric Association., 2013). Such diagnostic criteria are discussed in further detail later.

Social communication and social interaction deficits	• Difficulties with social-emotional reciprocity • Reduced sharing of interests and emotions • Difficulties in back-and-forth conversations • Difficulties with engaging in eye-contact • Difficulties understanding body language, gestures or facial expressions (non-verbal cues) • Relationship difficulties • Difficulties sharing imaginative play
Restrictive or repetitive patterns of behaviours, interests or activities	• Stereotyped or repetitive motor movements • Echolalia (automatic repetition of vocalisations made by another person) • Insistence on sameness and routines • Rigid thinking patterns • Difficulties with transitions or changes • Hyper- or hypo-activity to sensory input

Table 1: Some common behaviours frequently observed by autistic individuals in terms of the DSM-5 assessment criteria (adapted from Autism Speaks, 2013)

The classification of the autism acknowledges a vast heterogeneity of individuals, ranging from those with significant cognitive impairment to those with heightened cognitive abilities, compared to their neurotypical peers, as well as an often uneven profile of abilities across different domains (Aagten-Murphy et al., 2013; Agrawal, 2013; Chiang & Lin, 2007b; Fletcher-Watson & Happe, 2019; Whitby & Mancil, 2009a). Despite this heterogeneity, the underlying commonalities within this broad population revolve around difficulties with social communication and theory of mind (an awareness that others may think, operate or be motivated by different factors) and suggest some higher order cognitive process deficit, likely to be associated with weak central coherence (difficulties with global processing), discussed below (Williams et al., 2015). However, those individuals at the higher cognitive end of the spectrum, frequently have strengths in areas such as attention to, and memory based on, detail and a strong drive to detect patterns - 'systemising' - all of which are important skills within mathematics (Baron-Cohen, 2017; Kidron et al., 2018).

2.1.1 Debates over terminology and diagnostic criteria

In terms of diagnosis and access to educational support, although DSM-5 (American Psychiatric Association., 2013) is highly influential, the main criteria for diagnosing autism within the U.K. context, are those from the International Statistical Classification of Diseases and Related Health Problems (10th Edition) (ICD-10) (WHO, 2016). One key difference between these two diagnostic criteria is the categorisation of Asperger syndrome. Unlike DSM-5 (American Psychiatric Association., 2013), ICD-10 (WHO, 2016) still identifies Asperger syndrome as a separate condition, despite being under the umbrella term of autism. Within this definition, Asperger syndrome is characterised as a pervasive developmental disorder, differing from autism in the fact that there is generally no 'delay or retardation in language or cognitive development' (WHO, 2016, para. F84.5). Although the current study focuses on those individuals with a specific diagnosis of 'autism', rather than those with Asperger syndrome, it is important to acknowledge this label paradox, as many previous studies discussed within this literature review, refer to Asperger syndrome (and other subgroups, such as high functioning autism) as a separate condition. Those studies carried out before the introduction of DSM-5 (American Psychiatric Association., 2013), were indeed at a time when the two conditions were considered discrete in terms of diagnostic criteria and labels. Consequently, this paradox around labels and terminology, forms one of the main threats to the validity of some previous studies. The inconsistent and varied definitions used to describe and define this idiosyncratic spectrum condition, or the sub-groups within it, may result in the generalisability of the findings being somewhat questionable (Agrawal, 2013; Chiang & Lin, 2007; Mayes & Calhoun, 2006; Whitby & Mancil, 2009).

Autism, like many other disabilities or differences, is often viewed from two perspectives: the medical model of disability and the social model of disability (Anastasiou & Kauffman, 2011). Whilst this thesis does not debate the arguments of each model in-depth, it is important to acknowledge these differing perspectives. The medical model perceives disabilities, such as autism, as deficits, which need to be fixed through intervention or treatment. Considerable focus is placed on obtaining a diagnosis and treating the impairments. Critics of this model argue that the focus is on impairments or deficits, rather than considering autism from the interaction of neurological differences and the environment, thus focusing more on ‘fixing’ the individual, as opposed to ‘removing barriers and adapting the environment’ (Wallace et al., 2019, p. 8). In contrast, the social model considers such conditions based on individual’s strengths. Such diversity is valued as an asset to society and support is focused on the outcomes of harnessing the individual’s strengths.

Both DSM-5 and ICD-10 are frameworks for classifying health conditions, including autism, and are primarily based upon the [bio]medical model of disability (Bölte et al., 2018, p. 2), thus, the majority of research within this field is positioned from a medical model perspective. As a result, despite significant research into autism from a range of disciplines, our understanding still remains ‘largely disorder-focused’ (Baron-Cohen, 2017, p. 4). Indeed, DSM-5 omits to acknowledge some ‘implicit features of autism’, known to impact upon autistic individuals’ lives, such as masking behaviours, social anxiety and general anxiety, focusing only on those ‘impairments…identified at the point of diagnosis’ (Leatherland, 2018, pp. 27–28). Consequently, for those individuals, who have developed masking, or compensation strategies (Livingston et al., 2018)), it may be argued that such diagnostic criteria fail to represent their true personal experience of autism and may subsequently delay any formal diagnosis and associated access to appropriate support.

The argument around classification using the umbrella term ‘Autistic Spectrum Disorders’ (American Psychiatric Association., 2013) surrounds ‘the term ‘disorder’ - implying [that] the natural order has gone awry and that the individual’s underlying cognition and neurobiology is dysfunctional is some way’ (Baron-Cohen, S., 2017, p. 744; Fletcher-Watson & Happe, 2019). However, Baron-Cohen (2017) highlights the fact that in some circumstances and contexts, those individuals identified as having this ‘disorder’ can indeed function comparable to, or at a higher level than, their neurotypical peers. Furthermore, such diagnostic criteria, which often ‘present deficit-focused checklists of impairments’, runs the risk of ‘denying [autistic individuals] with the benefit of a positive self-identity (Leatherland, 2018, p.31) and is often ‘characterised against a backdrop of presumed normative standards’ (Fletcher-Watson & Happe, 2019, p. 44).

Supporting this debate within the use of diagnostic criteria and normative standards, Milton (2012) suggests the double empathy problem: whilst autistic individuals may present difficulties in understanding behaviours and actions of the NT population, the reverse may apply too (Chown, 2017). Milton (2012) suggests that a consequence of this may be that for autistic individuals to adapt into society, they must learn to understand society, whereas, when viewed from the opposite perceptive, this does not apply. The result of this double empathy problem leads to ‘autistic people often developing a greater understanding of the ways in which non-autistic people interact that non-autistic people develop of autistic ways of interacting’ (Chown, 2017, p.167).

Rather than considering ASD as a disability, as recognised by the U.K. government, who describe autism as a ‘hidden disability’ (as it is not always obvious what the individual’s specific needs actually are) (APPGA, The National Autistic Society, 2017, p. 10), Baron-Cohen (2017) suggests the use of the alternative term ‘neurodiversity’. This term, aligning more closely with a social-model perspective, is perhaps better placed to capture the variation in cognitive abilities within this population, as opposed to applying a disabling societal label. He argues that the term ‘disability’ assumes that there is nothing positive about the condition and that it relies upon societal acceptance and support. However, when co-morbidity with other conditions exists, such as epilepsy, critics dispute the use of the term neurodiversity. Such critics suggest that a disability or dysfunction is indeed present in these individuals (Baron-Cohen, 2017), as there is a condition present, which requires [medical] intervention or treatment. Nevertheless, it could be argued that those autistic individuals with co-morbid epilepsy, for example, may be both neurodiverse and have a disability, depending on one’s perspective. Moreover, such co-morbidity with other conditions is often the focus of research within this population, particularly research into therapies to support these conditions (Wallace et al., 2019). Consequently, the current study excludes those individuals with co-morbid conditions to include only those individuals with a diagnosis of autism, to address this gap in the research.

Whilst the debates over terminology and perspectives (medical versus social) continue, the current study also establishes a consistent approach to referring to this population. As such, it draws upon findings from a recent online survey carried out in the U.K. (Kenny et al., 2015) as a justification for the use of 'autistic' [person], rather than [person] 'with autism'. From the survey (N=3470, age range 19-66+ years), 'autistic' was the preferred term for autistic individuals or family members (N=2361), rather than 'with autism', which tends to be the preferred term amongst professional communities (N=1109). Findings from the qualitative aspect of this study, suggest that a key driver in this preferred language choice is the favour of separation of an individual's autism from their identity, in conjunction with the focus on autism being seen as negative, rather than acknowledge the 'positive characteristics of autism' (Kenny et al., 2015, p. 457) – more closely aligning with a social model viewpoint. Based on this research, the current study uses the term 'autistic', when referring to this population.

Whilst acknowledging the heterogeneity of autism, various key theories have been proposed to explain and understand the social and cognitive difficulties faced by many individuals within this population. To provide further understanding of some of these associated behaviours and difficulties, perspectives based on the two models (medical and social) are considered in turn. Each of the models can be used as a basis for considering some of the current theories, which attempt to provide an explanation for such cognitive and social behaviours and difficulties. Nevertheless, the 'usefulness' of any of the theories presented should be questioned, in terms of supporting the development in the understanding of autism for both neurotypical individuals and for autistic individuals themselves (Leatherland, 2018, p.39).

2.1.2 Autism from a medical model perspective: Cognitive theories

Three key theories underpinning cognition and autism, based on the medical model of disability are: theory of mind deficit (ToM); theory of executive dysfunction (EF); and weak central coherence (WCC) theory, all of which may provide some explanation to some of the commonly presented factors associated with autism and cognition.

Considered more from medical model perspective of autism, as opposed to a social model, these cognitive theories may also provide some explanation for the social behaviours commonly observed or experienced within this population. Each of these three theories is discussed in turn below.

Theory of Mind Deficit

One major theory associated with autism that has dominated the field for many years, concerns a deficit in theory of mind (ToM). A deficit in ToM may be defined as a failure in the understanding of, and thinking about, other people's thoughts and feelings' (Wittemeyer et al., 2011, p. 11), or 'mentalising' (Fletcher-Watson & Happe, 2019, p. 68) and may go some way to explaining many of the social and cognitive deficits commonly observed and experienced within the autistic population.

In support of ToM deficit, various studies have suggested that autistic individuals in general, perform less well on ToM tasks and tests, when compared to their neurotypical (NT) peers (Berenguer et al., 2017). ToM is commonly tested for with unexpected transfer tests of false belief, a psychological test in which individuals are measured on their cognitive ability to attribute false beliefs to others (Fletcher-Watson & Happe, 2019).

The lack of awareness that others may think, operate, or be motivated by different factors, goes a long way to explaining some of the difficulties around social awareness and social communication experienced by many individuals on the autistic spectrum. Subsequently, the commonly observed lack of understanding of verbal- and non-verbal social cues, difficulties in providing appropriate responses within social situations and the literal interpretation of language (particularly figurative language) can be explained through the mechanism of ToM deficit. In support of this, several previous studies have concluded that ToM does indeed play a key role in, and an adequate explanation for, some of the difficulties in social development and competence of some autistic individuals, as well as for the difficulties with social behaviours frequently observed (Kimhi, 2014; Mazza, Mariano, Peretti, Masedu, Pino,

& Valenti, 2017). However, as Fletcher-Watson & Happe, (2019) point out, just because autistic individuals may experience difficulties understanding other people's thoughts or actions, 'emotional empathy' is a separate skillset, therefore such difficulties do not suggest that these individuals do not care (p.72).

Whilst it is often claimed that autistic children have difficulties in engaging in 'pretend-play', which is often seen as a consequence of ToM deficit, it is evidenced that such individuals are equally as capable of this activity than their language-matched controls, they may simply choose not to do so (Boucher, 1989; Levy, 2007). This choice not to engage in pretend play may be due to a lack of understanding of the fictional situation or simply down to a lack of motivation or self-satisfaction in the activity.

Despite the study by Berenguer et al. (2017) claiming a clear discrepancy in performance between autistic individuals and neurotypical (NT) individuals on measures of ToM, the theory itself does not adequately account for the range of difficulties universal in autism (Chown, 2017), such as restricted and repetitive interests, or monotropism (Murray et al., 2005) (discussed in chapter 2.1.5) and cognitive aspects such as generalisation of understanding to new tasks. Furthermore, it does little to support the understanding of the mechanisms underlying some of these commonly observable behaviours and differences seen amongst the autistic population. Coupled with this lack of explanation of all associated difficulties and differences, is the heterogeneity of autism as a spectrum condition – the range of individuals classified within this spectrum is vast, and consequently, all individuals within this population may not necessarily display difficulties with ToM. Adding to this debate, it is also suggested that problems with ToM have been reported in other clinical groups, such as those individuals who have suffered strokes, schizophrenic individuals, those with dementia, those with bipolar affective disorder and deaf-born children (Chown, 2017; Fletcher-Watson & Happe, 2019).

In addition to 20% of students with autism passing the basic tests of ToM (Happé, 1999), a previous study, which focused on those individuals with Asperger Syndrome, concluded that 73% of these students passed not only the basic tests, but also more advanced tests of ToM (Bowler, 1992). Within Bowler's (1992) earlier study, two control groups were included: one made up of socially impaired chronic schizophrenic subjects (N=15) and another of 'non-handicapped' subjects (N=15) (p.879). It was considered that the first control group, who were subjects often displaying later-onset social impairment, represented a useful comparison with those individuals in the Asperger group (N=15) in terms of the development of social impairment. Despite the higher figures of individuals passing the ToM tests, the author of this study also attempted to 'maximise the possibility of revealing impaired abilities' (Bowler, 1992, p. 881), through the application of strict criteria with which to 'pass' the test. Furthermore, 73% of the individuals with Asperger's syndrome were also able to correctly answer questions relying on second-order theory of mind ('John thinks that Mary thinks…' (p.882)). This finding was deemed to be significantly different to the findings from both control groups (67% for schizophrenic group and 80% for the non-disabled control group), indicating that ToM deficits are not necessarily universal across the autistic population and would not be a sole indicator of the condition.

However, the minimal variation between the groups in this study may be influenced by the age of the participants (mean age of Asperger's group = 26.67 years), which is notably higher than many previous studies carried out on autistic groups to test ToM. It is suggested that 'an individual's ToM ability may, ay least partially, 'catch up' with that of their TD peers during adolescence and young childhood' (Chown, 2017, p. 146). Thus, as the sample consisted of adults in this study, this may suggest that they have developed the ability to develop non-social strategies in order to provide correct responses to the false-belief questions, or may have developed their ToM ability further. Berenguer et al. (2017) suggest that sometimes this may be the case, as adults have had a longer period to develop these strategies, and therefore inevitably have more experiences to draw upon in order to develop such non-social strategies.

Nevertheless, despite developing (non-social) strategies to enable them to correctly answer false-belief questions, thus seemingly masking ToM deficits, it is suggested that they may have difficulties generalising the solutions to real life (Berenguer et al., 2017), which may be linked to weak central coherence (discussed below). As a result, this may impact upon their ability to extrapolate relevant data and visually represent and solve real life mathematical word problems within the social context of the classroom.

Contrary to the indication that ToM may not provide an explanation of the behaviours associated with autism, and may indeed be widely varying in occurrence within this population, Levy (2007) proposes that:

1. 'Autistic individuals have an impaired ability to attribute mental states to others;
2. that this is caused by some specific impairment of higher-order representational capacity;
3. that this specific impairment, when fully understood, will be seen to be primary, in the sense of not itself needing further behavioural explanation and being able to explain all the criterial features of autistic behaviour.'
(Levy, 2007, pp. 860–861)

What is unclear in Levy's (2007) proposition is what the 'specific impairment of higher-order representational capacity' (p.861) is, and furthermore, how this may give rise to the other 'criterial features of autism.' Additionally, such criterial features of autism are also subject to debate, depending on which criteria are referred to for the diagnosis, as discussed above.

The studies presented in this section indicate ambiguous support for the potential significance of ToM deficits amongst the autistic population. Whilst it appears that ToM deficits may indeed provide an explanation for some of the behaviours associated with this population, in some cases, the evidence is inconclusive in providing an explanation for all the commonly associated behaviours and the heterogeneity of this population. Consequently, this leaves researchers with the challenge of finding a suitable definition of autism, which can adequately explain all the underlying mechanisms behind the wide range of associated behaviours commonly presented.

Theory of Executive Dysfunction

Whereas cognitive functions are the 'tools' and concern what knowledge, skills or intellectual equipment a person possesses, executive functions (EFs) can be described as the 'processes' that comprise how a person executes these cognitive abilities (Roelofs et al., 2015, p. 126) and 'operate across both social and non-social contexts' (Fletcher-Watson & Happe, 2019, p.94). The EFs are associated with the prefrontal areas of the frontal lobes of the brain and include behaviours such as planning, impulse control, inhibition of prepotent responses (being able to suppress alternative, often more motivating or intrinsically rewarding responses), set maintenance (maintaining the focus of attention of the information provided), organised search and flexibility of thought and action' (Goldstein & Naglieri, 2014; Ozonoff et al., 1991).

Consequently, deficits in the EFs may go some way to explain those behaviours commonly associated with autism, not accounted for by ToM, such as the need for sameness, difficulty switching attention and the lack of impulsive control (Levy, 2007, pp. 231–232). The 'executive functions' refer to a set of higher order, cognitive processes associated with control, directing and monitoring thoughts, actions and goal-directed activities (Lecce et al., 2019; Wen, 2018; Ziermans et al., 2017). The EFs encompass a broad spectrum of interrelated processes that are implicit in guiding, directing, and controlling cognitive, planning, working memory, emotional and behavioural functions. Such processes are most apparent during the active solution of novel problem (Berenguer et al., 2017; Gioia et al., 2000; Goldstein & Naglieri, 2014) and are suggested to aid the retrieval of learned information (Ullman & Pullman, 2015) thus may be significant in mathematical word problem solving.

Of particular significance to the current study, a metanalysis of 64 articles, suggests a global impairment in working memory to be common amongst the autistic population (Desaunay et al., 2019). Furthermore, it is suggested that working memory capacity, which encompasses both visuo-spatial and verbal processing, may be closely linked to academic attainment – particularly in mathematics, where the correlation appears to be strong across the age phases (Wen, 2018). Whilst the EF skills are considered interrelated, and ‘represent a single, latent construct in pre-school children’ (Lecce et al., 2019, p. 2), they can indeed be distinguished from one another, and it is suggested that as children progress beyond early childhood, this construct indeed becomes ‘fractionated’ (Keenan et al., 2019; Wen, 2018). The rapid development of EF skills appears to coincide with the transition from early childhood to formal schooling, and consequently, it is thought that such EF development is facilitated through the structure usually found in the formal classroom environment (Keenan et al., 2019). Specifically, it is suggested that working memory capacity increases significantly between the ages of 6-16 years, with pupils aged 10-11 years performing significantly better on working memory tasks than those aged 8-9 years (Wen, 2018). The current study draws on this fractionation of the construct in terms of data collection and analysis through obtaining data relating to specific behaviours linked to the range of EF skills (discussed in chapter 3). Furthermore, the role of epistemic emotions – both positive (such as enjoyment, pride, self-perception) and negative (such as boredom, fear of failure) – may influence processes such as self-regulated learning, motivation and task initiation and engagement, subsequently impacting upon levels of achievement (Gravemeijer, 2020; Muis et al., 2015). Consequently, within the current research, pupils’ self-perception and enjoyment of mathematical problem-solving is considered, as this may play a key role in the executive functions within the participants.

In terms of providing an explanation for the rigid and repetitive behaviours and cognitive inflexibility, the theory of deficits in EFs provides some explanation for this, in terms of giving rise to impairments in the initiation of new actions and difficulties in evading learnt patterns of behaviour (Hill & Frith, 2003; Keenan et al., 2019; Roelofs et al., 2015). Whilst there is a lack of agreement within the literature on the specific EF impairments associated with, and consistent with autism, poor cognitive shifting and control - the ability to shift between two different thoughts, actions, processes or sets, accordingly to changes in the situation (Hill, 2004; Wen, 2018), has been identified as an EF impairment specific to autism (Ozonoff et al., 1991; Roelofs et al., 2015; Wen, 2018).

According to Wen (2018), shifting comprises two components:

1. the formation of a mental set, which requires the individual to discriminate between the relevant and irrelevant information;
2. the ability to switch to a new set, potentially conflicting with the previous set.

Thus, whilst not necessarily displaying impaired cognitive functions, difficulties in cognitive shifting may inevitably impact upon an individual's ability to function adequately in everyday life, particularly within the context of the classroom (Roelofs et al., 2015).

Consequently, when it comes to solving mathematical word problems (discussed later), any impairment in an individual's ability in set-shifting, may impact upon their ability to interpret and execute a range of mathematical problems.

In terms of implications within the classroom, EF skills require selection of appropriate cognitive strategies, coupled with careful monitoring of the effectiveness of the task throughout to enhance the accuracy and efficiency with which knowledge is acquired and problems are solved (Bae, 2013; Goldstein & Naglieri, 2014). Additionally, commonly associated problems seen in individuals who have impaired EF skills, may include disorganised actions and strategies for everyday tasks; difficulties with decision making; and selective filtering of information (Goldstein & Naglieri, 2014). Deficits in these skills of planning and decision making may have clear cognitive implications, particularly when it comes to solving mathematical word problems (Kintsch & Greeno, 1985; Polya, 1945), as discussed later in this chapter. Such deficits will inevitably impact upon students' ability in making choices as to the identification of the appropriate operations, along with the selection and construction of relevant visual representations required to accurately represent the relationships between the information presented within the word problem. Furthermore, Keenan et al. (2019) suggest that the ability to 'selectively ignore information' and to 'complete tasks whilst retaining verbal instructions' is a requirement of successful learning (p.2). Therefore any such impairment within the EF skills (particularly set maintenance, working memory and inhibitory control – the 'suppression of prepotent responses' (Wen, 2018, p. 17)) may impact on pupils' ability to execute mathematical learning tasks. In fact, it is suggested that the EF skills of working memory and inhibitory control may provide a more accurate predictor of mathematical achievement, than IQ (Alloway & Alloway, 2010). Considered from a wider perspective, the ability to devise and execute an overall plan through which to process the word problem through to the solution phase (Polya, 1945), may be significantly restricted if these EF skills are impaired.

However, like the ToM hypothesis, EF does not provide an explanation and understanding of all the difficulties and behaviours commonly associated with individuals on the autism spectrum. EF difficulties are not unique to those with autism, especially 'intellectually high functioning individuals' (Chown, 2017, p.180), they are common amongst with other conditions (which also happen to be common co-morbid conditions with autism, such as attention deficit hyperactivity disorder (ADHD) and obsessive-compulsive disorder (OCD)). Again, due to the overlap with co-morbid conditions, the current study utilises participants with a sole diagnosis of autism to ascertain the significant conditions potentially influencing mathematical problem solving associated with autism, rather than other [co-morbid] conditions.

Nevertheless, in a comparative study, when it came to assessing the EF skills, the proportion of autistic subjects performing worse than the NT control group mean was 96% (Ozonoff et al., 1991, p. 1094). However a similar subsequent study suggested that this was indeed only 50% (Pellicano, Maybery, et al., 2006), emphasising the heterogeneity of autism as a spectrum condition.

This significant variation in the findings of these two studies may be as a result of the 'liberal [inclusion] criterion' applied in Ozonoff's (1991) study (Pellicano, Maybery, et al., 2006, p. 92). Whereas in Pellicano et al.'s (2006) study, the age range of the participants in the autistic group (N=40) was 4-7 years (mean 5.6 years, S.D. 0.91), Ozonoff's (1991) study included autistic participants (N=23) with an age range of 8-20 years (mean 12.05, S.D. 3.19). The mean age in Ozonoff's (1991) study was therefore more than double that of Pellicano et al.'s (2006) study, and furthermore, the large standard deviation suggests a greater spread of ages. Thus, the two studies are not comparable in terms of the participants included, and therefore, caution should be applied when comparing the findings of these two studies as the impact of age cannot be accounted for. Additionally, Pellicano et al.'s (2006) study only included individuals with a verbal and non-verbal IQ measure of 80 or more and excluded individuals with any co-morbid conditions. On the contrary, Ozonoff's (1991) study included individuals with an IQ of 69 or greater and consisted of 17 individuals with a diagnosis of autism (according to DSM-3) and 6 with a diagnosis of pervasive developmental disorder not otherwise specified (PDD-NOS). It is thus not surprising to find that Pellicano et al.'s (2006) results were noticeably lower than those presented by Ozonoff (1991), due to the variability in inclusion criteria applied to the participants within each study and the findings are therefore specific to a specific subgroup of this population. However, in terms of IQ, there is significant debate over the reliability of traditional measures, which may not capture an accurate result for the autistic population. Traditional measures for the construct of IQ may not fully capture autistic individual's areas of strength and may therefore imply a significant underestimation of their actual intelligence (Wallace et al., 2019). Therefore, the use of IQ as a construct to measure an autistic individual's intelligence should be considered with caution within empirical studies.

In summary, whilst difficulties in the EFs can offer some potential explanation for some of the cognitive and motor difficulties common amongst autistic individuals, like other theories, it also indicates significant EF profile variation amongst this population. Due to this variation, the current study includes data collection relating to specific behaviours associated with the EF skills (discussed in chapter 3.4.) in an attempt to capture this variation and potential fractionation of EF skills (Keenan et al., 2019; Lecce et al., 2019; Wen, 2018).

Theory of Weak Central Coherence

The theories presented thus far – ToM and EF, have 'struggled to provide a complete account' of the autism spectrum and its wide range of associated behaviours and impairments (Pellicano, Maybery, et al., 2006, p. 77). It is argued that, although these theories go some way to provide an explanation for some of the behaviours associated with autism, they may not provide an adequate explanation for the "islets of ability" frequently observed within the autistic population. When considering those individuals with 'good visuospatial ability, enhanced rote memory, and an uneven IQ profile', those autistic individuals 'exhibit a peculiar style of information processing' (Pellicano, Maybery, et al., 2006, pp. 77–78), with a 'local processing bias' (Booth & Happé, 2010, p. 377). This unusual processing style, common amongst the autistic population, is referred to as weak central coherence (WCC), hence giving rise to the theory of WCC.

Central coherence can be defined as the 'in built propensity to form meaningful links over a range of stimuli and to generalise over as wide a range of contexts as possible' (Aljunied & Frederickson, 2013, p. 172), and is often diminished (or weak) in individuals with autism. Central to this theory is the 'need to integrate information, which is variously described as top-down processing, global processing, parallel processing, processing wholes, or integrating information in context' (Bae, 2013, p. 51). Rather than processing concepts and problems at a global level, as commonly seen amongst NT individuals, autistic individuals have a tendency to process concepts and problems 'in a detail-focused way, processing constituent parts, rather than the whole' (Levy, 2007, p. 237), consequently ideas may become 'detached from their context' (Aljunied & Frederickson, 2013, p. 173). One explanation for this is that these individuals may possess visual attention that becomes spatially confused, giving rise to the processing of local, rather than global, information (Anderson, 2019; Harré & Madden, 1975; Levy, 2007; MacLeavy, 2019). However, rather than being viewed as a deficit (despite the use of the term 'weak' potentially suggesting this), WCC is frequently referred to as a 'cognitive style' (Booth et al., 2003, p. 392) or a 'processing bias towards features, rather than a processing deficit for wholes' (Happé & Frith, 2006, p. 14). Viewed from this perspective, WCC may be considered as an extreme end of a ‘normal continuum of cognitive style’ (ibid. p.16).

Nevertheless, since WCC was first discussed by Frith (1989) as a focus on local, rather than global processing, subsequent research has extended this theory further. Consideration is now given as to whether observable behaviours resultant of the theory of WCC are indeed a consequence of a reduced ability in global processing, a heightened ability in local processing, or indeed a combination of the two (Booth & Happé, 2018).

WCC may go some way to explaining the frequently observed correlation between good reading accuracy but poor text comprehension in autistic individuals, supporting the concept of the theory that strengths lie in the local detail, rather than making sense of the global meaning. In addition, WCC may explain the apparent ‘superiority’ of autistic individuals on tasks which require local processing, and the difficulties seen within this population on tasks requiring the ‘integration of information in context’ (Pellicano, Maybery, et al., 2006, p. 78). In support of this, Desauney et al. (2019) suggest that WCC may provide an explanation for the frequently observed difficulties in drawing on different elements of information to form a coherent, relational representation.

Cognitively, this frequently results in those individuals with WCC presenting with difficulties in transferring learning from one context to another (Aljunied & Frederickson, 2013; Booth & Happé, 2010). However, as with other cognitive theories in autism, research findings suggest a lack of consistency of WCC across the autistic population. Some critics have argued against the validity of central coherence as a construct and there is inconsistency in the findings of studies into the weaknesses in central coherence when compared to neurotypical control groups on tasks requiring local processing (Pellicano, Maybery, et al., 2006).

Findings from a study in Singapore into the effects of dynamic assessments on central coherence (N=52, mean age = 9:10 years), indicate that autistic individuals with high general intellectual ability tended not to show WCC (Aljunied & Frederickson, 2013). Although a control group was not used within this study to allow the findings to be compared to the NT population, these findings do align with other studies, which suggest significant cognitive variability amongst this population (Pellicano, Murray, et al., 2006). However, the authors suggest that those individuals with higher general intellectual ability may be more adept at developing compensatory strategies to overcome the difficulties, thus potentially masking any WCC. Nevertheless, this provides further evidence that the cognitive profiles of autistic children show significant variation.

Evidence from previous studies suggests that WCC may be independent to, and possibly underpin, the theories of ToM and EF, thus indicating the role of central coherence as an explanation for associated behaviours seen within autistic individuals (Booth & Happé, 2010; Happé, 1994; Jolliffe & Baron-Cohen, 1999). Corroborated by these studies, findings from the research carried out by Booth et al. (2003), which set out to examine whether WCC occurs due to difficulties with EF, concluded that any impairments in the ability to plan tasks (EF) did not account for any tendency towards a focus on detail (WCC). Additionally, when analysing the findings from the TD group in the study, they found that individuals who were poor at planning, were less likely to focus on details. As such, their conclusion from these findings suggest that any dysfunction in the EFs (e.g. the planning of tasks), does not provide an explanation for WCC (the focusing on detailed aspects of the task). Therefore, the study claims that 'detail focus (WCC) is a characteristic of autism unrelated to impairments in executive skills (planning)' (Booth et al., 2003, p. 392).

In contrast, it may be that EF provides an explanation for WCC due to the difficulty in shifting between local and global processing (EF) explaining difficulties in global processing (Happé & Frith, 2006). One task designed to identify local and global processing tendencies is the stem sentence completion task (Booth & Happé, 2010). When carried out with NT control groups (N=176), ASD groups (N=41) and a group comprising individuals with a diagnosis of attention deficit hyperactivity disorder (ADHD) (a condition known to lead towards a bias for local processing) (N=29), the authors' results showed a significant bias towards local processing in both the ASD and ADHD group, when compared to the NT group. When comparing the local responses of the ASD group to the NT group, the mean completion score for the ASD group (n=41) was 15.63, compared to 17.47 for the NT group (aged 8-10 years, n=47) and 17.38 (aged 11-13 years, n=40). Note, the NT group were split according to age ranges. In addition to these findings, which support the high occurrence of WCC within ASD individuals, the authors also suggest that performance on this task was not related to inhibitory control, thus arguing that WCC is a separate cognitive deficit to EF, supporting previous studies (Booth et al., 2003).

Results from Pellicano's (2006) study go some way to support Booth et al.'s (2003) theory, suggesting 'evidence of local processing on coherence tasks to be widespread' (WCC) amongst autistic individuals (N=40), when compared to a matched control group (N=40) (matched on age, gender, verbal and non-verbal ability). However, performance on false belief tasks (ToM) and aspects of EF were less widespread within this group. The study was focused on tasks requiring visuospatial coherence and was the first study of its kind to explore the functioning of autistic individuals across all three areas (WCC, ToM and EF) within the same sample.

Overall, the study's findings suggest that children with ASD demonstrate better performance than typically developing (TD) matched controls, on tasks requiring local processing. However, they are 'less adept on tasks requiring visuospatial integration', supporting previous theories of profiles of strengths and weaknesses in different cognitive aspects (Pellicano, Maybery, et al., 2006, p. 95). Furthermore, and perhaps more significantly, the study also concluded that 'several core underlying deficits co-exist within ASD, as opposed to one [single] deficit' (p.95).

Whilst it is apparent that these cognitive theories do not independently adequately account for all of the commonly observed behaviours associated with autism (neither do they claim to do so), they each propose some explanation as to some of the behaviours commonly seen within this population. These theories provide some explanation for many of the cognitive difficulties and impairments, often associated with autistic individuals and thus primarily construct autism from a medical model perspective.

2.1.3 Autism from a social model perspective

In addition to the cognitive theories discussed above, an alternative social-model approach, to attempt to identify different clusters of individuals within the spectrum, is proposed through the consideration of 'profiles of autistic difficulties' (Siegel, 2009, p.2). When viewed from this social model perspective, the spectrum condition can be viewed as 'clusters of autistic disabilities' (Siegel, 2009, p.2). Based on this perspective, individuals' behaviours within the spectrum may be described in terms of their 'cluster deficits': social cluster deficits; nonverbal communication cluster deficits; verbal communication cluster deficits and play and exploration cluster deficits (Siegel, 2009).

According to Siegel (2009), deficits within each of these clusters, may give rise to the following patterns of behaviour and cognition:

- *Social cluster deficits:*
    - *Lack of desire to please others; instrumental and incidental learning;*
- *Nonverbal communication cluster deficits:*
    - *Anxiety; difficulty following and understanding nonverbal cues and gestures;*
- *Verbal communication deficits:*
    - *Difficulties with language processing; limitations to pure/working memory;*
- *Play and exploration cluster deficits:*
    - *Difficulties with schema development; aversion to novelty; limited learning through exploration (which may be argued as more cognitive than social deficits).*

By considering the associated deficits within these clusters, the varying levels of underpinning cognitive theories also become apparent, particularly ToM. Those individuals with social deficit clusters may display a lack of awareness of others and a lack of socio-emotional reciprocity (Wittemeyer et al., 2011), in line with aspects of theory of mind. When considering non-verbal communication deficits, limited comprehension and use of facial cues, along with lack of awareness of gestures, may also align with theory of mind. In addition, with verbal communication deficits, both receptive and expressive language may be impaired, and with play and exploration deficits, a lack of imaginary play and repetitive behaviours may be observed (Siegel, 2009) – all more closely aligned with specific ToM deficits (Boucher, 1989; Levy, 2007), rather than EF or WCC.

Unlike the cognitive theories, these cluster deficits do not directly take account of specific cognitive deficits (although there is clearly overlap). However, they may go some way to help educational researchers, and consequently, may more effectively address the issue of heterogeneity within the autism spectrum when it comes to application of appropriate pedagogical skills through considering the autistic population within these clusters.

From an educational perspective, individuals within these clusters can be offered appropriate pedagogical strategies to support these clustering deficits through drawing out common themes within the clusters based on the varied cognitive profiles of pupils on the spectrum. Furthermore, in an attempt to consider autism from a wider social and educational perspective, the International Classification of Functioning, Disability and Health (ICF) have considered a series of core sets which clearly identify specific behaviours associated with autism, in support of more accurate diagnosis. These core sets contain behaviours based on the age of the individual, encompassing environmental factors, to provide a diagnostic tool, which goes beyond a medical model viewpoint (Bölte et al., 2018). The aim of these core sets is to 'mark a milestone toward the standardized assessment of autism spectrum disorder–related functioning in educational, administrative, clinical, and research settings' (ibid. p.1). It is anticipated that these core sets, along with influence from DSM-5 (American Psychiatric Association., 2013), will be used to support the development of the newest version of the International Classification of Diseases (ICD-11), which will consider the implications of health conditions upon individuals' functioning (Bölte et al., 2018). Nonetheless, the core set for school age children (6-16 years) still contains 81 categories, which although may provide some useful markers for educationalists and researchers, still includes a wide range of skills. Within this set, many educationally relevant and personal development skills can be found, for example problem-solving, listening and developing relationships (p.13). The list of skills within this set encompass both cognitive and social development and may provide a useful tool for curriculum design and assessment in terms of educational settings.

2.1.4 Monotropism theory

Whilst the theories discussed thus far have struggled to provide a suitable explanation for the heterogeneity of observed and experienced behaviours commonly associated with autism, one theory may offer a more comprehensive understanding of the spectrum: monotropism. Developed by autistic academics (Fletcher-Watson & Happe, 2019), according to Murray et al. (2005), this theory is 'central to the autistic condition' and constitutes 'atypical patterns of attention' (p.139). Building on from the theory of WCC, discussed above, it is suggested that the difficulties often experienced by autistic individuals may not be purely down to a tendency towards local processing, rather than global processing, but rather a consequence of difficulties relating to attention and task demand (Murray et al., 2005). Monotropism theory proposes that, on the basis that a limited amount of attention is available to any individual at any point, this will be focused on 'having few interests highly aroused' (Murray et al., 2005, p.140). In contrast, when considering the general population, a normal distribution, in terms of the spread of attention, generally gives rise to 'many interests less highly aroused – polytropic tendencies' (ibid. p.140). Therefore, when considering WCC, Murray et al. (2005) propose that 'rather than preference for local processing, over global processing' it is more likely that there is a 'tendency towards hyper-awareness within the attention tunnel and a hypo-awareness outside of it' (p.142), explaining the restricted and repetitive interests. Therefore, such 'monotropic tendencies' (Murray et al., 2005, p.140) offer an explanation for the commonly experienced ‘restricted and repetitive behaviours and interests’ (Bölte et al., 2018, p. 1) associated with autistic individuals, and which form part of the diagnostic criteria for autism within DSM-5, and are suggested to be central to autism Chown, 2017; Murray et al., 2005). Such a cognitive style (monotropism) may also explain some of the EF deficits, discussed earlier, such as attention and set-shifting (Hill, 2004; Wen, 2018), as these functions ‘all require broadly distributed attention (polytropic tendencies) (Leatherland, 2018; Murray et al., 2005). Furthermore, in addition to offering an explanation of many characteristics of autism, monotropism theory makes theoretical assumptions, based on the personal accounts and experiences of autistic individuals, providing a theory which draws upon the lived experience of autistic individuals themselves (Leatherland, 2018; Milton, 2012; Murray et al., 2005), which according to Fletcher-Watson & Happe (2019), should be an integral aspect in any high quality theory of autism. In contrast, those cognitive theories (ToM, EF and WCC, discussed above), are based on observed behaviours of autistic individuals, taking no account of the autistic voice, or ‘subjective experience’ (Leatherland, 2018, p.59).

When considering the frequently observed, uneven profile of abilities across different domains (Aagten-Murphy et al., 2013; Agrawal, 2013; Chiang & Lin, 2007; Whitby & Mancil, 2009) amongst the autistic population, this theory may go some way to explain this. One consequence of monotropic tendencies is ‘a very fragmentary view of the world’, which leads to ‘an uneven skills profile’ (Murray et al., 2005, p.148). Furthermore, another outcome of monotropism may be a lack of awareness of ‘other people’s viewpoints’ (ibid., p.148), therefore supporting those characteristics accounted for through ToM.

This section has provided a further theory to explain the characteristics often experienced amongst autistic individuals, with an emphasis on developing a theory based on the autistic voice (Milton, 2012). Monotropism may provide an explanation for many of these common characteristics and may support a better understanding of autism for the neurotypical population, as well as for autistic individuals themselves.

2.1.5 Mathematical profiles of autistic students

The following section begins to develop a deeper understanding into the mathematical profiles of autistic individuals. Whilst some scholars highlight mathematical abilities amongst the autistic population as a commonality (Iuculano, 2012), data from several studies suggest that mathematical profiles amongst this population, like the spectrum itself, are considerably varied (Aagten-Murphy et al., 2015; Benaron, 2009; Iuculano, 2012; Keen et al., 2015; Wei et al., 2015). Through drawing on empirical studies and research literature, this section develops an understanding into some of the factors and mechanisms underlying mathematical abilities within this population, which are then used to guide the conditions for analysis within the current study. Evidence is drawn from a range of empirical studies, many of which focus on specific sub-groups.

A study on Cambridge undergraduates found that 1.85% of mathematics students were officially diagnosed with autism (N=378) compared to just 0.24% of students from other disciplines (N=414), leading to the claim that mathematical talent (which is based on systemising) is linked to autism (Baron-Cohen et al., 2007). All those mathematics students with an official diagnosis of autism were male. However, the proportion of males in the mathematics group (74%) were significantly higher than in the other disciplines (39%), which may somewhat skew these findings, as the prevalence of autism in males is significantly higher than in females within this study. Furthermore, as the study only included students at Cambridge University, this impacts upon the generalisability of these findings.

In a review of literature relating to the academic achievement of autistic individuals, Keen et al. (2015) found significant variability in the findings. Whilst IQ appeared to be predictive of academic achievement in reading, group means for achievement in mathematics were average or below average for those pupils with IQ>70 (Keen et al., 2015). Due to the frequency of co-morbid intellectual disabilities and the wide IQ range associated with this population, some studies have shown that as many as 16% of autistic individuals may display ‘hypercalculia’ (where the mathematical ability of an individual is more advanced than the individual’s general learning ability overall). In contrast to this, other studies suggest that 20-25% of these individuals have mathematical learning difficulties, often with number, calculation and problem solving, which may be indicative of developmental dyscalculia (Aagten-Murphy et al., 2015; Gevarter et al., 2016; Root et al., 2019). However, when it comes to savant skills, it is estimated that only 10%, or less, of autistic individuals may display these (Benaron, 2009). Savant skills refer to a cognitive phenomenon, where an individual demonstrates extraordinary skills, rarely seen amongst the general population. Such skills generally tend to occur in one or more of five major areas of art, memory, arithmetic, musical abilities, and spatial skills, thus may include hypercalculia. With respect to memory, whilst that associated with general knowledge appears to be very strong, episodic memory (the capacity to encode personally experienced memories which occurred only once), appears to be significantly diminished within the autistic population, although tends to show an increase from ages 8-9 years (Desaunay et al., 2019; Wen, 2018).

There is limited empirical research in the field of academic achievement and strategies to support the most cognitively able autistic students and it is currently unknown whether existing interventions are appropriate or effective for those with higher cognitive abilities (Assouline et al., 2012; Doobay et al., 2014). Furthermore, the lack of research carried out in school settings and involving data collected from school personnel, rather than standardised tests alone, is also very limited (Keen et al., 2015). Consequently, the nature the current study aims to address this gap in research, through utilising primary data collected from classroom teachers.

A two-group comparison study (N=40), on a subgroup of the autistic population, carried out by Bae (2013) concluded that the overall problem-solving ability of pupils with high-functioning autism (HFA) (IQ>80, N=20), aged 9-12 years, was significantly lower than that of typically developing peers, matched on age and IQ (N=20). The definition of HFA used within Bae's (2010) study, which is no longer seen as a separate diagnosis under current criteria (discussed above), was based on those autistic pupils with an IQ>80 and scoring >28 on the Childhood Autism Rating Scale – High Functioning Version (CARS) (Schopler et al., 2010). Problem-solving ability was measured using the TOMA2-SP test and the Mathematical Word Problem Solving (MWPS) test, which are norm-referenced and criterion-based tests respectively. Within the HFA sub-group, the mean problem-solving score was 9.35 (S.D. 4.06) and 12.75 (S.D. 8.46) compared to 12.55 (S.D. 2.76) and 20.15 (S.D. 3.18) for the typically developing group (TOMA2-SP test and MWPS test respectively), indicating a notable difference in problem solving ability between the two groups. In addition, supporting the findings from other studies on this population where the variability of results is substantial (Chiang & Lin, 2007; Keen et al., 2015; Mayes & Calhoun, 2006; Whitby & Mancil, 2009), the standard deviations of the HFA subgroup were substantially greater than for the typically developing group. Aligning with several other studies focusing on reading fluency, decoding and mathematical attainment (Björn et al., 2016; Boonen et al., 2013; Fuchs et al., 2006; Jones et al., 2009; Kintsch & Greeno, 1985; Kleinert et al., 2015; Oswald et al., 2016; Powell et al., 2019; Wei et al., 2015; Whitby & Mancil, 2009), Bae's (2013) study suggested stronger correlations between solving word problems and word reading and decoding for the autistic group when compared to the TD group. When further analysis was carried out controlling for other significant factors in each group, everyday mathematical knowledge appeared to be the consistently significant factor accounting for the differences in word problem solving ability between the two groups (Bae, 2013). Additionally, abilities in mathematical vocabulary and computation appeared to positively correlate with mathematical word problem solving ability. This corroborates Fletcher and Santoli's (2003) research, who suggest that a focus on the teaching of mathematical vocabulary may improve students' mathematical performance and thus, claim that ability with vocabulary may be synonymous with ability in problem solving in students with autism (Fletcher & Santoli, 2003). Bae et al*.,* (2015, p.2206) suggest that 'with appropriate instruction, students with autism may have the potential to perform as well academically as their neurotypical peers.' They go on to propose that 'teachers should apply appropriate instructions to assist them in improving reading comprehension, math vocabulary, computation, and everyday mathematical knowledge' (ibid. p.2206).

Further research indicates a positive correlation between expressive communication, reading ability and mathematical achievement amongst the autistic population (Björn et al., 2016; Boonen et al., 2013; Jones et al., 2009; Kintsch & Greeno, 1985; Kleinert et al., 2015; Oswald et al., 2016; Powell et al., 2019; Wei et al., 2015; Whitby & Mancil, 2009). However, it is also likely that similar correlations may also be seen within the NT population. Although one limitation of much previous research into the relationship between reading and mathematical performance is that it has primarily focused on reading fluency, rather than text comprehension, the study by Björn et al. (2016), specifically looked at the correlation between text comprehension and mathematical word problem solving. It is suggested that the cognitive skills required in reading comprehension, including the role of the executive functions and working memory, may be very similar to those required in mathematical problem solving (Björn et al., 2016; Özsoy, 2015). Consequently, any underlying difficulties in reading comprehension (and, based on this finding, the EFs) may give rise to difficulties with mathematical word problem solving.

In their longitudinal study with Finnish students, Björn et al. (2016) also found gender to be significant – text comprehension skills in boys in 4th grade (age 10-11 years but in their 4th year of compulsory schooling) predicted mathematical word problem solving skills in 7th grade (age 13-14 years but in their 7th year of compulsory schooling). Contrastingly, in girls, text comprehension in 4th grade, appeared to be a predictor of mathematical word problem solving performance in grade 9 (ibid.). However, Bae's (2013) study suggested that the correlation with solving word problems against sentence comprehension (rather than word decoding and fluency) and mathematical knowledge for the ASD group was much weaker, when compared with the TD group.

In contrast to findings from a previous study carried out by Fuchs et al. (2006), which suggested a correlation between reading fluency and mathematical word problem solving ability, the study by Björn (2016) found reading fluency not to be a predictor of mathematical word problem solving ability, particularly in subsequent years. This discrepancy may be due to a number of factors, one of which the authors identify as the use of phonological decoding as a proxy for phonological processing (Fuchs et al., 2006). It is also worth acknowledging that language itself may be a factor, due to the differences in the phonological regularity between Finnish and English languages, hence influencing pupils' overall fluency. A further possible explanation for the differences may be because in the former study, the sample was selected based on a test of computational fluency, which Bae (2013) suggests being influential on problem solving ability. In contrast, the sample from the latter study was taken from a heterogeneous group of all children born in 1993 within proximity to a specific Finnish school district, perhaps giving rise to a more diverse range of participants.

An argument made that threatens the validity of some studies is over the use of standardised tests, such as The Wechsler Scale (Wechsler, 2003). It is argued that these standardised tests do not necessarily 'assess students' abilities to apply basic mathematical knowledge in insightful ways' (Chiang & Lin, 2007, p. 553) and the range of scores within this population is often significantly broad (Griswold et al., 2002). Such findings contribute to the concept of the broad spectrum of autism and further complicates empirical research involving this diverse population. Additionally, difficulties around expressive and receptive language may compromise measures of conceptual understanding, as this is 'frequently inferred through verbal justification and reasoning' (Donlan et al., 2007, p. 25), as evidenced in the study by Agrawal (2013). However, some studies indicate that those individuals with a diagnosis of Asperger Syndrome, scored significantly higher on the verbal comprehension index than those pupils with a diagnosis of autism (Doobay et al., 2014; Foley-Nicpon et al., 2012). Whilst this finding may be significant for future research, it must also be considered that the changes in current diagnostic criteria (discussed above) may make this difficult to generalise. For those individuals receiving their diagnosis based on DSM-5 (American Psychiatric Association., 2013), Asperger syndrome is no longer considered a separate diagnosis from autism.

Viewed from a different perspective, research by Temple Grandin, a highly acclaimed academic on the autistic spectrum herself, suggests that 'three different specialised cognitive types exist amongst those individuals with autism/Asperger Syndrome: visual thinkers; pattern thinkers; and verbal specialists' (Grandin, 2009, p. 1437). Consequently, these cognitive types may influence the academic strengths and weaknesses of these individuals. For example, Grandin (2009) suggests that pattern thinkers tend to excel at mathematics and music, however, have difficulties with reading and writing composition, whereas visual thinkers often struggle in mathematical areas such as algebra. She suggests that concept formation is often based on categorisation of specific visual memories in visual thinkers and argues therefore that problem solving abilities can be increased with ‘exposure to more and more experiences’ (p.1437). This may have significant implications for educators to be aware of the cognitive type of the individuals within the classroom. She goes on to suggest that ‘word-based tasks are processed in the visual part of the brain for those individuals with autism/Asperger Syndrome’ (p.1439), aligning with findings from a previous study which also identified the involvement of language in mathematical cognition (Donlan et al., 2007). As the ability to solve real-life mathematical word problems is likely reliant upon, and linked to, the ability to comprehend the word-based problem (Björn et al., 2016; Özsoy, 2015), these findings may be significant to educators. It is essential that those supporting these individuals understand the potential barriers to word-problem solving ability amongst this population and explore appropriate teaching and learning strategies to support them in reaching their potential in this area of mathematics.

Gandin’s (2009) personal account provides a useful insight for researchers in the field, as it considers the perspectives of the research subject, which is rarely the case in much empirical research in this area. However, it is also very specific to the individual and must be acknowledged within the vast heterogeneity of this population. Nevertheless, Grandin’s (2009) perspectives do align with other empirical research into the role of visual processes and cognitive abilities, in that strengths or deficits in this area may have implications on the mathematical ability of those individuals with autism/Asperger Syndrome (Assouline et al., 2012; Bae, 2013; Chiang & Lin, 2007; Whitby & Mancil, 2009).

When analysing the use of visual representations in supporting autistic pupils with mathematical problem solving, the findings are mixed. Bae (2013) identified that only 3/20 HFA pupils and 4/20 typically developing pupils made use of visual representations within their problem-solving approach. Bae (2013) concluded from this that the use of visual representations is not associated with the word problem solving abilities of autistic pupils [and likewise, as the data would indicate, for typically developing pupils]. The potential explanation for this finding is that the ASD pupils may not have been able to represent the problems in a visual way, possibly due to difficulties with the 'integration and processing of the linguistic content of the word problems' (Bae, 2013, p. 111).

Nevertheless, the research is pertinent to the current study, as providing further evidence for the apparent gap in problem-solving abilities between autistic pupils and their typically developing peers. This builds upon findings from an earlier study in the U.S. (N=21) which suggests that individuals with Asperger Syndrome frequently have diminished ability to problem solve. It has been suggested that in order to support these pupils, who often have difficulties organising their thoughts, visual representations (discussed in more detail later) can be a useful tool for supporting with understanding numerical operations and concepts (Griswold et al., 2002, p. 99). It is clear from the studies by Bae (2013) and Griswold (2002) that there are mixed findings as to the usefulness of visual representations to support autistic students with mathematical problem solving. It is also suggested that there is a need for further research into the specific factors associated with the problem-solving process of autistic pupils, as this will support the development of effective instruction for this population (Bae, 2013; Chiang & Lin, 2007).

A significant impairment often seen within the autistic population, according to Williams et al. (2015), is flexibility in abstract reasoning, more so than any impairments in rule learning, supporting the theory of WCC. Consequently, it is suggested that such individuals may demonstrate underlying difficulties with conceptual organisation and thus, whilst often being able to 'apply previously identified classification rules', may experience more difficulty with the formation of new categories (p.861). This suggests that the application and acquisition of new information, which may not fit into existing schemas (discussed further in chapter 2.2.2), may be an underlying difficulty for these individuals. The findings from this study claim to be relevant to individuals across the cognitive range of the spectrum, however a suggestion of the need for further consideration to problem solving skills is made by the authors. It should also be noted that this research focused on adolescents and adults, therefore there is no evidence to suggest that these findings can be generalised to children of primary school age.

2.1.6 Summary

Based on the literature discussed thus far, it seems clear that no single theory of autism (medical, cognitive or social) can adequately and confidently account for, and explain, all of the social behaviours and cognitive difficulties commonly associated with the autistic population. However, what this perhaps does suggest, is the understanding of autism as an 'umbrella term' for a multitude of different 'conditions', or an explanation as to the heterogeneity of the observed behaviours and differences common amongst this population.

The cognitive and social theories presented above, go some way to provide an explanation for some of the behaviours associated with autism, whilst also illustrating the heterogeneity of the spectrum and difficulty in establishing a clear operational definition of autism for the purposes of empirical research. Due to the complexities of providing an over-arching explanation for the mechanisms underlying the frequently observed range of cognitive difficulties, it could be argued that when analysing a range of studies, the participants may not be comparable. The individuals within each of these studies, whilst being autistic, may present very different abilities and difficulties.

Consequently, autism is often considered through multiple-deficit accounts, recognising that individuals may experience differing degrees of difficulty in each of these areas, accounting for the complexity of autism (Levy, 2007). Furthermore, it acknowledges that rather than having clear boundaries, autism encompasses a wide cognitive heterogeneity of individuals. Little research has been conducted, which specifically focuses on the 'variables mediating the deficits' commonly seen amongst the autistic population; instead, has tended to focus on the difficulties accounted for by specific theories (Berenguer et al., 2017, p. 430).

Adding to the complexities of autism even further, it is suggested that some autistic individuals may indeed 'compensate' for difficulties in cognitive areas, such as ToM, consequently concealing such underlying cognitive difficulties through observable behaviours (Livingston et al., 2018; Ullman & Pullman, 2015). Livington et al.'s (2018) study found that those individuals who demonstrated a high ability to compensate for ToM difficulties, tended to show higher IQ and EF skills and vice-versa, although WCC did not appear to be affected. Consequently, such 'compensation' strategies, which the authors of the study refer to as a 'mechanism' (ibid. p.1), add a further complexity to understanding and identifying the underlying causal mechanisms behind the observed behaviours associated with autism. Although, as Livingston et al. (2018) suggest, further research is needed within this area, particularly longitudinal research, these findings may go some way to provide an explanation for the uneven cognitive profiles and the 'islets of ability' (Pellicano, Murray, et al., 2006) seen within this population.

Due to the complexity of autism, coupled with the lack of clear boundaries to define this umbrella term, a clear definition of autism is required. Consequently, based on the literature discussed above, the following definition of autism spectrum disorder is used within this study:

Autism is a neurodiverse condition, affecting social communication, covering a wide spectrum of behaviours, resulting in an often very uneven cognitive profile of individuals. Commonly, underpinning the difficulties and 'islets of ability', lies an unusual information processing system, frequently resulting in weak central coherence. Based on the uneven profiles amongst the autistic population, four core clusters of observable behaviours can be used to classify individuals on the spectrum, based on complexities within common themes: social skills; non-verbal communication; verbal communication; and play and exploration, although there may inevitably be overlap amongst these clusters.

Overall, this section has provided an overview of the literature relating to the underlying explanations for the mechanisms and associated behaviours related to the cognitive and social difficulties commonly seen within the autistic population. The studies discussed provide mixed evidence for the support of any individual theory to provide an adequate explanation, which confidently accounts for all associated deficits and profiles seen across the spectrum. However, there appears to be more consistent evidence presented for the role of central coherence underpinning autism and the use of ‘clusters of autistic disabilities’ may provide one option for helping to classify individuals within the spectrum into more meaningful groups, through drawing out common themes, in terms of exploring the effectiveness of specific pedagogical approaches for this population.

Due to the heterogeneity of individuals on the autistic spectrum, it is challenging to find specific groups and factors, which may be directly linked to academic achievement within this population (Keen et al., 2015; Wei et al., 2015) and more specifically, a detailed understanding of the variability within mathematical achievement of this group (Aagten-Murphy et al., 2013; Keen et al., 2015). Whilst factors such as mathematical vocabulary, reading fluency, reading comprehension and the use of representations, all appear potentially influential in autistic pupils’ mathematical problem-solving ability, the research findings are somewhat mixed.

Evidence has been presented as to the potential range of factors, which may influence mathematical ability and problem-solving ability within individuals on the autism spectrum. Furthermore, differences between sub-groups of autistic individuals in terms of their mathematical performance begin to emerge. Many previous studies have focused specifically on those individuals classified as HFA or Asperger’s Syndrome, however, there is a distinct lack of evidence from studies identifying key conditions associated with mathematical problem-solving and the variation across the wider spectrum, which this study hopes to address.

Through drawing on the findings from the literature above, combined with those from the subsequent sections of the literature review, the current study seeks to identify the potentially significant conditions[4] associated with mathematical problem-solving performance for this population. Through considering the literature, a conceptual framework, on which the current study is based, is developed (see Chapter 2.3 below). This conceptual framework provides a firm underpinning for the research design and process, which seeks to explore the conditions, or combinations of condition, which may be necessary, or sufficient[5], for autistic pupils to reach the correct solution to mathematical word problems.

Drawing on the literature from this section, the key factors considered to be significant in mathematical problem solving for autistic pupils are represented through the following initial conceptual framework, which is further developed throughout the literature review:

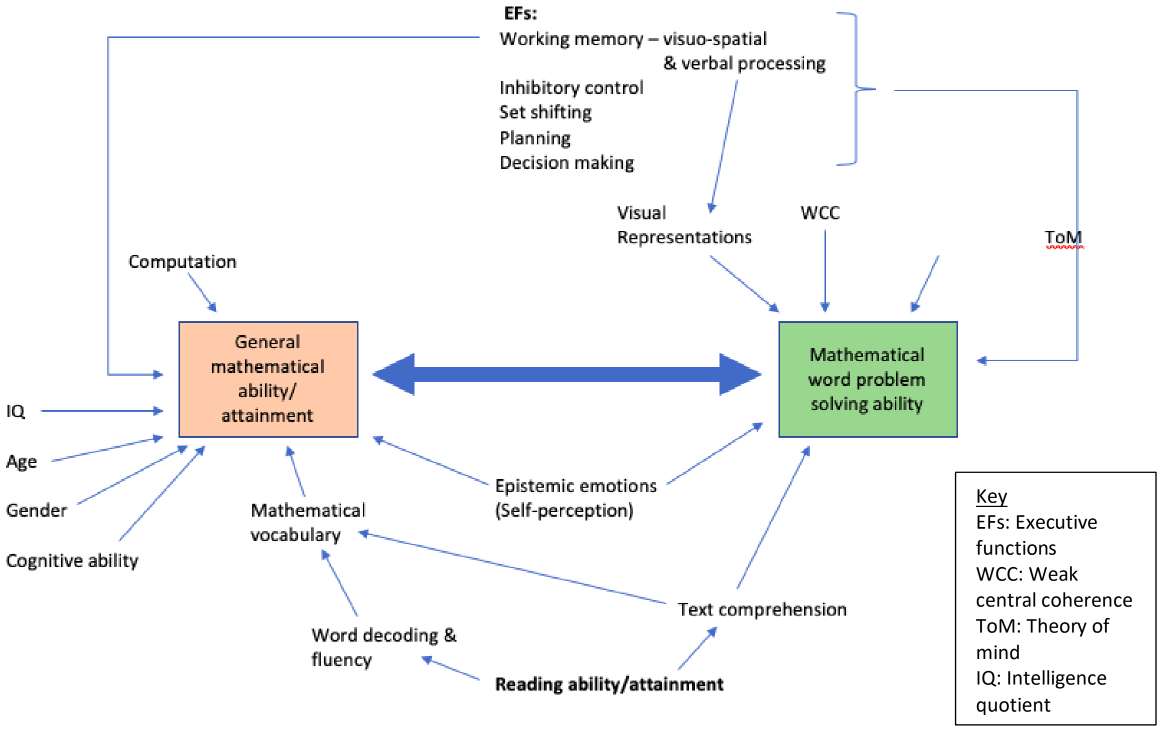


Figure 1: The key conditions deemed to be potentially significant in mathematical problem solving for autistic pupils

[4] As discussed in the introduction, the term 'conditions' is synonymous with factors. As the current study is based on the methodological framework of Qualitative Comparative Analysis (QCA), the terminology of 'condition' is applied. A condition refers to a factor, which gives rise to the a desired, and specified, outcome.

[5] The terms 'necessary' and 'sufficient' are used here in line with the methodological approach of QCA and refer to the same concepts explained within the introduction.

2.2 Mathematical problem solving

This section explores some of the current and previous research around problem solving, specifically relating to mathematics. Through reviewing the literature, an understanding of the meaning and conceptualisation of problem solving is established. An exploration of some of the key theoretical frameworks for problem solving are presented, before moving on to examine mathematical word problems as a specific type of problem solving. Following this, 'real-life' word problems are discussed, with an emphasis on the linguistic styles of such genre of word problem, along with the implicit classroom pedagogies and cultures often underpinning mathematical problem solving within the school context. Finally, visual representations within mathematics are explored, before culminating in the theoretical underpinnings and development of the bar model, as a visual representation within mathematical problem solving.

According to Alan Schoenfeld (1983), most 'mathematical problems' encountered in the elementary (primary) school classroom, are indeed exercises, and 'unless the context is unusual, it is not problem solving' (p.41). However, this argument might be extended to consider a condition where the context was usual, but the solution strategy not so, thus perhaps still constituting problem solving. This argument builds on his prior work, where he discusses problem solving skills and the requirement to be able to draw upon, and adapt, techniques from previous tasks to produce 'plausible approaches' to be applied to unfamiliar problems (Schoenfeld, 1982, p. 43). The focus on problem solving as a 'skill' to be 'learned' is stressed by Pratt and Woods (2007, p.82) where an emphasis on learning and knowledge acquisition supports pupils in decontextualizing knowledge into a form that is required within the classroom.

Since the 1980's, research into problem solving has shifted its focus more towards the inclusion of cognitive behaviours, such as self-monitoring of progress and self-reflection on performance (Kilpatrick, 1985) – aspects which may be considered as the executive functions. Here, the potential interplay between cognitive difficulties faced by autistic pupils (discussed above) and mathematical problem solving begin to emerge. As educators, it is essential that consideration is given to how children learn and develop their knowledge (social and cognitive psychology) within the boundaries of specific disciplines (mathematics), therefore, it could be argued that 'mathematics education may serve as the mediator between cognitive psychology and mathematics' (Schoenfeld, 1983, p. 43), which this study will explore further. It is suggested that the ability to solve problems in real-life is not necessarily synonymous with the ability to solve mathematical problems in the classroom (Gerofsky, 1996). From a cognitive psychology perspective, 'learning (and problem-solving) takes place when students construct new mathematical knowledge by reflecting on their physical and mental actions' (Gningue et al., 2014, p. 161). This builds upon Bruner's (1966) theory of modes of representation, where, as children learn, they move through three distinct modes of representation: enactive (requiring concrete materials), iconic (the creation of mental representations) and symbolic (manipulation of symbols) (Gningue et al., 2014).

Kilpatrick (1985, pp.7-8), suggests that the solution of complex problems requires:

1. A rich store of organised knowledge about the content domain;
2. A set of procedures for representing and transforming the problem;
3. A control system to guide the selection of knowledge and procedures.

These three requirements, align with the formation and access to schemas and storage and retrieval of information from the short-term and long-term memory (1); the acquisition and utilisation of visual representations (2); and control of the executive functions (3) – all of which are discussed within this chapter.

When it comes to knowledge, Mayer (1985) suggests that this constitutes 'linguistic and factual (content knowledge), schematic (knowledge of problem type), algorithmic (knowledge of computational procedures), and strategic (knowledge of when to apply procedures and how to represent problems)' (pp.123-138). Consequently, it is important that classroom pedagogy pays consideration to these, in addition to 'when and how to utilise such knowledge' (Lester, 1985, p. 43). This builds upon Schoenfeld's (1983) work, where he suggests the core concept behind problem solving is decision making (Fülöp, 2019), which may link to the EFs (discussed above). The core decision-making processes involved in mathematical problem solving are represented in figure 2 below:

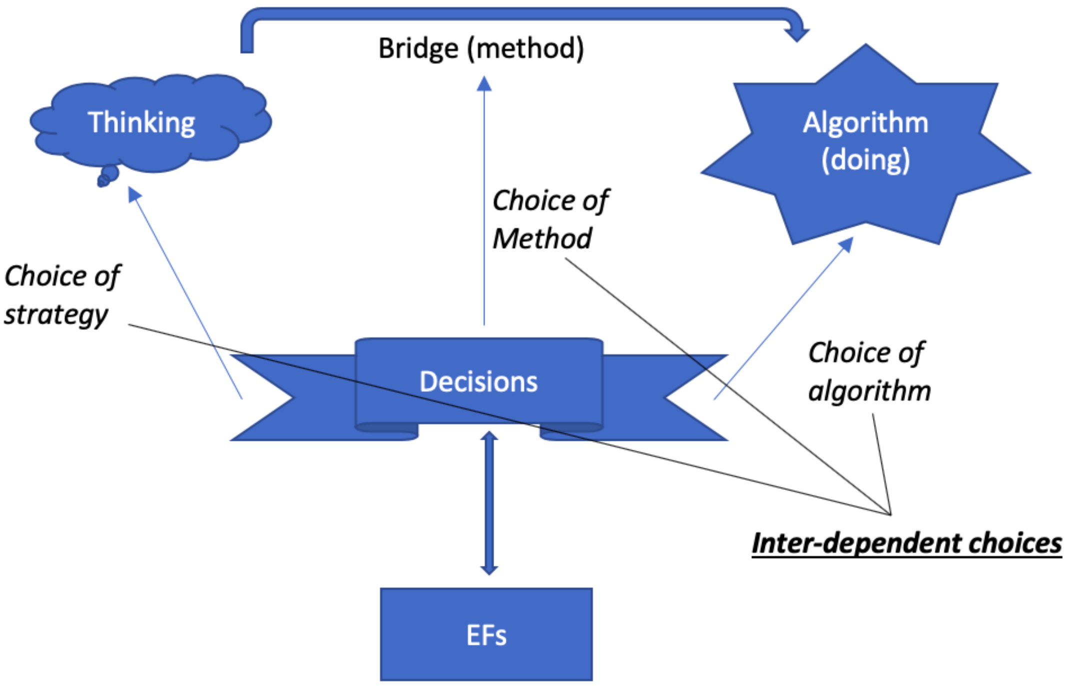


Figure 2: Levels of decision making within the mathematical problem-solving process (adapted from Fülöp, (2019))

Drawing on the literature discussed above, for the purposes of this study, the following definition is used to operationalise the concept of mathematical problem solving:

'A task presented to students in an instructional setting that poses a question to be answered but for which the students do not have a readily available procedure or strategy for answering it' (Lester & Cai, 2016, p.117).

Through reflecting on the pedagogical implications discussed above (Fülöp, 2019; Kilpatrick, 1985; Lester & Cai, 2016; Mayer, 1985), and the role that the demands on working memory may play within the problem-solving process, it is apt to pay brief consideration at this point to cognitive load theory (Sweller, 2011), before moving on to discuss some of the relevant theories of problem solving.

2.2.1 Cognitive load theory

Cognitive load theory is an instructional design theory based on our knowledge of, and aspects of, human cognitive architecture (Leahy & Sweller, 2011; Sweller et al., 2019). In simplistic terms, the theory suggests that new information is first processed by a capacity- and duration-limited working memory, before being stored for later use in the (unlimited) long-term memory (Sweller, 2011). Within cognitive load theory, knowledge is considered as biologically primary or biologically secondary. Primary knowledge being that which has evolved and been acquired over a considerable amount of time and is unconsciously obtained. Biologically secondary knowledge is determined by our culture and is characteristically the subject of information taught within the school classroom (Leahy & Sweller, 2011; Sweller et al., 2019). According to cognitive load theory, new information is initially stored in the short-term memory, which has a limited capacity. Only once information is embedded within the long-term memory, can it be retrieved for later use and drawn upon to solve unfamiliar problems. In terms of classroom practice, it is important to consider how information is presented, to reduce the load of pupils' working memory and to utilise the relevant information stored in the long-term memory (biologically secondary knowledge). The processing of novel information is limited by pupils' working memory capacity, whereas the processing of familiar information from the long-term memory is not limited (Sweller et al., 2019). Therefore, it may be argued that pupils' 'performance on complex, cognitive tasks', is dependent upon 'whether the amount of information presented is equal to, or greater than, the available working memory' (Ngeno et al., 2019, p. 254). When considering the implications on working memory deficits, discussed above, there may be significant implications of diminished EFs, particularly working memory, for some autistic pupils. The access to, and construction of new schemas within the problem-solving process, may impact upon pupils' cognitive load and consequently, when one's working memory capacity is reached, performance in managing and processing any additional information may be impaired. Consequently, the probability of errors occurring is increased (Kalyuga, 2013; Ngeno et al., 2019).

In terms of mathematics-specific learning, the implications of instructional design based on cognitive load theory have been disputed due to the nature of the development of mathematical knowledge (Watson, 2019). Whereas cognitive load theory suggests instruction based on the presentation of limited new information, in mathematics, it has been argued that ‘new’ mathematical ideas are often extensions of familiar ideas and concepts, which are retrieved from the long-term memory (Watson, 2019). Therefore, according to Watson’s (2019) critique, the construction of mathematical concepts over time is less related to working memory, as this is not ‘new’, unconnected material. Nevertheless, working memory capacity is deemed to play a significant role in the acquisition of new information, therefore if a pupil does not have the required information accessible within their long-term memory, then the acquisition of new information (mathematical or not) will inevitably be hindered (Rogers, 2017). Furthermore, it is suggested that other factors, such as stress, emotions and uncertainty may indeed impact upon working memory capacity, as these factors may be in competition with task retrieval processes (Sweller et al., 2019).

When considering those autistic individuals who may be utilising ‘compensation’ strategies (Livingston et al., 2018), this may in fact lead to an increase in cognitive load. It is suggested that compensation may add to the cognitive load, particularly if such a strategy draws upon cognitive resources, which are specific to this process and thus not necessarily utilised within NT peers. Consequently, such enhanced cognitive load may ‘be prone to breakdown when these resources are depleted’ (Livingston et al., 2018, p. 5), thus revealing true cognitive ability when accessing tasks, such as mathematical word-problem solving, which already add to cognitive load.

2.2.2 The role of schema in mathematical problem solving

Many previous and current theories of problem solving incorporate aspects of mental representation and the use and development of schemas, which utilise, and are often dependent on, an individual’s long-term and working memory capacity (Fuchs et al., 2006; Kintsch & Greeno, 1985; Mayer, 1989; Peled & Wittrock, 1990; Schoenfeld & Herrman, 1982; Schoenfeld, 2018). Although schemas tend to be stored in long-term memory, rather than the working memory, the nature of the schema size, and perhaps more importantly the enactment of a schema, may be reliant on working memory capacity. In turn, any diminished capacity in working memory (discussed earlier) may have implications when it comes to mathematical problem solving.

According to Marshall (1995), a schema is defined as 'a memory structure that develops from an individual's experiences and guides the individual's response to the environment' (p.15). When faced with a new problem, students then either draw upon an existing schema in to classify the problem according to those requiring a similar solution path or begin to develop a new schema based upon a novel problem type.

Further developed by Kintsch and Greeno (1985), in their model for solving mathematical word problems, Schoenfeld and Herman (1982) discuss how an individual's problem schema is accessed when perceiving a problem. Additionally, Mayer (1985) suggests that to understand a problem, the ability to build a schema (if an existing one is not already in place) for that type of problem presented may be required. This suggests either a 'straightforward' response, if accessing an existing problem schema, or a 'less automatic response' if the development of a new schema is required (Schoenfeld & Herrman, 1982, p. 484). It is suggested that students may attempt to find the 'appropriate schema after reading only a few words', which may lead to errors from the outset (Mayer, 1985, p. 128). Based on this argument, it is advocated that how an individual perceives a problem (which it is suggested is influenced by the ability to form a mental representation of the problem) is therefore a 'crucial component of problem solving success' (Schoenfeld & Herrman, 1982, p. 484).

Bearing resemblance to cognitive load theory, Kintsch and Greeno (1985) discuss the need for pupils to hold the set schema information within their 'short-term buffer memory' and utilising their 'episodic memory' (which they suggest is not necessarily separate from other memories) (p.120). The short-term buffer memory has a limited capacity, and thus influences the available resources for maintaining further information or carrying out subsequent calculations required for the solution of the word problem. Any dysfunction in the EF skills may impede the rehearsal process required in the utilisation of short-term memory (Desaunay et al., 2019).

Schema theory, proposes that 'interconnected pathways within the brain are used to process and categorise new information' based on existing schemas (Maglicco, 2016, p. 23). Through considering cognitive load theory (Sweller, 2011; Sweller et al., 2019), the cognitive demand is thus reduced if students are able to draw upon, or relate to, existing schemas of knowledge, rather than having to store additional information in their short-term memory. This theory aligns closely with the first of Polya's (1945) problem solving steps (understanding the problem), which is discussed below.

With these theories in mind, and through drawing on a question as a worked example, schema development, along with its impact on cognitive load, may be represented as follows:

Joe has three marbles. Tom has five marbles. How many marbles do they have altogether?
(Kintsch & Greeno, 1985, p. 113).

Although it could be challenged as to whether this question constitutes a word problem, or merely a calculation, we can draw on Boonen's (2013) work to justify this. Discussed in more detail later, Boonen (2013) suggests that a word problem can simply be considered as 'any mathematical exercise where significant background information on the problem is presented as text, rather than in mathematical notation' (p.271). Therefore, based on both Boonen's (2013) definition, and that of Lester and Cai (2016), the above question can be considered a word problem.

Joe has three marbles (S1) → Short-term buffer memory

Tom has five marbles (S2) → Short-term buffer memory ((S1) displaced into episodic memory)

How many marbles altogether (S3) → (S1) retrieved from episodic memory

(S2) held in short-term buffer memory

In line with cognitive load theory, Kintsch and Greeno (1985) propose that as the size of the information within the sets (S1, S2, S3), required to be held in the short-term buffer memory, increase, the likelihood of reaching the correct solution for the word problem, decreases (p.122). However, contrary to this, we can return to consider Watson's (2019) critique of cognitive load theory within mathematics (discussed above). In this example, it could be argued that if the question is presented as written text and the calculations are worked out in a written format, then very little of this information is required to be held in the short-term buffer memory.

When considering the cluster deficits explored earlier (Siegel, 2009), individuals with verbal communication deficits may be particularly prone to difficulties within this area due to limitations of short-term memory capacity. In addition, individuals with play and exploration cluster deficits, may display difficulties with the development of new schemas, thus potentially influencing their word problem-solving ability (Siegel, 2009a). At this point, it becomes clear to see how cognitive load theory (Leahy & Sweller, 2011) and schema theory (Maglicco, 2016), may be significant for autistic pupils with respect to poor episodic and working memory (Desaunay et al., 2019).

In support of Schoenfeld and Herrman's (1982) line of reasoning, it is suggested that an individual's mental representation of a problem (which is examined in further detail within chapter 2.2.8 on visual representations), is constructed through the following process:

1. The recall or generation of a problem-solving schema;
2. Relation of the schema to the problem;
3. The generation of possible meanings to the problem.

(Peled & Wittrock, 1990, p. 176)

Peled and Wittrock (1990) also discuss the 'coherence' of problems, in relation to the correspondence between the problem and the schema. They suggest that 'some problems are coherent with obvious relations to schema', whereas others may be 'incoherent without any obvious relations or schemas, which must be generated' (pp.166-177). Furthermore, the likelihood of reaching a successful solution is determined by how closely the representation of the problem and the mathematical structure within the problem align, or if an existing schema is readily available (Maglicco, 2016; Peled & Wittrock, 1990).

2.2.3 Models of mathematical problem solving

One model of problem solving, which builds upon the use of schemas as an essential aspect to success within this domain, is proposed by Mayer (1989). His two-phase model of problem solving (specifically relating to mathematical word problem-solving), is based upon two key theories, considered to be central to his model: schema theory (discussed above) and problem-solving theory (Mayer, 1989).

Problem solving theory is based wholly upon Polya's (1945) four-stage process to solving word problems:

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Look back

According to research in the 1980's, the intention of Polya (1945) in his development of this process, was to 'uncover useful strategies for inquiry into novel situations'

(Schoenfeld, 1983, p. 43) and ‘many of Polya’s heuristic suggestions are ‘metacognitive prompts’’ (Kilpatrick, 1985, p. 10). The pedagogical focus of Polya’s (1945) problem solving theory is to draw attention to pupils’ actively making sense of problems through their engagement in the mathematical activity of problem solving (Pratt & Woods, 2007). The four steps involved in Polya’s (1945) problem solving, all draw upon skills relating to the EFs (Wen, 2018) (discussed earlier), and may therefore be significant for those autistic individuals where these are diminished.

Through combining schema theory and problem-solving theory, Mayer (1989) developed his two-stage model of problem solving – the problem representation stage and the problem solution stage. Within this model, it is proposed that students will use their existing schemas used to solve previous problems in their solution of a novel problem. However, for those individuals who lack such schema, the likelihood of correctly representing the novel problem accurately, is reduced.

Within the problem representation stage, the text from the word problem is converted to an internal representation – ‘building blocks of mental models constituting students’ content knowledge of a particular topic’ (Rau, 2017, p. 720), through drawing on students’ existing schemas, before being translated into an external representation by drawing on reading comprehension skills and schematic knowledge. Students then act upon this representation during the problem-solving stage through application of appropriate algorithms, interpreted within the context of the problem. This algorithm requires students to choose the operation, calculation strategy and computation skills required to solve the problem and correctly represent the solution within the appropriate context of the problem (Maglicco, 2016).

When considering the theoretical underpinnings of word problems, coupled with the operationalisation of the bar model, as a visual representation, Mayer’s (1989) problem-solving theory is further developed. Mahoney (2012) draws upon Mayer’s (1989) work, in conjunction with schema theory and problem-solving theory, in the development of a theoretical framework underpinning the bar model. This framework is discussed further in chapter 2.2.9, where the bar model, as a visual representation to support mathematical problem solving, is discussed in detail.

In terms of 'competence' in problem-solving, Goldin (1987) proposes five 'higher-level languages':

1. A verbal/syntactic system;
2. Non-verbal systems for imaginistic processing;
3. Formal notational systems of representation;
4. A system for heuristic planning and executive control;
5. An affective system which monitors and evaluates problem-solving progress.

When considering Goldin's (1987) higher-level languages, it is clear to see the role of EFs in four and five above, along with the potential relationships to Siegel's (2009) cluster deficits associated with autism (1 – verbal communication cluster; 2 – non-verbal communication cluster; 3 – play and exploration cluster). Hence, in terms of problem-solving competence, the role of attributes common amongst autistic individuals may well be significant.

When considering the theories discussed thus far, we can begin to consider the pedagogical implications for the teaching and learning of problem-solving skills within the classroom.

2.2.4 Pedagogical Implications in Problem-Solving

When considering the pedagogical implications associated with problem solving, it is pertinent to consider Skemp's (1978) theory of relational and instrumental understanding. Relational understanding refers to the ability to correctly apply procedures and techniques across a range of contexts and problems, whereas instrumental understanding refers to the use of written procedures, rules or tools. This theory can also be aligned with conceptual and procedural understanding – conceptual relating to relational understanding and procedural relating to instrumental understanding (Mutawah et al., 2019). Consequently, in the following section, the terms are used interchangeably – relational and conceptual understanding, and instrumental and procedural understanding.

Relational understanding enables students to draw upon, and reflect upon, their mental representations and modify these previously known ideas, to apply to a new context (Gningue et al., 2014). Furthermore, such [conceptual] understanding relates to pupils' ability to reason and comprehend concepts in mathematics, as well as to understand the relationships between such concepts and operations, thus supporting pupils with solving non-routine problems (Gningue et al., 2014; Rittle-Johnson et al., 2001). In addition, it is suggested that a secure conceptual understanding may support pupils to avoid errors relating to magnitude – a key factor is the correct application of the bar model (discussed later) (Mutawah et al., 2019). Thus, according to Gningue et al.'s (2014) conceptualisation of relational understanding, any inhibition on pupils' schema access and development is likely to be detrimental to the evolution of this type of understanding. Likewise, for those individuals with WCC, the ability to generalise and modify such prior knowledge to new (unfamiliar) contexts, is likely to be impaired. Again, when considering the cognitive theories and cluster models explored earlier, the potential influence of WCC and deficits within the play and exploration cluster, appear to be potentially significant factors in determining the overall word problem-solving ability.

According to Rittle-Johnson et al., (2001), procedural understanding is defined as 'the ability to execute action sequences to solve problems' (p.346), similar to Skemp's (1978) instrumental understanding. This ability to plan and execute the steps within the problem-solving process relies heavily upon the EFs, and as a result, any impairment in the EFs, may significantly impact upon this stage of word problem-solving (Roelofs et al., 2015). However, instrumental understanding may provide difficulties for students if new experiences are encountered that don't fit their existing schemas (Maglicco, 2016; Peled & Wittrock, 1990; Skemp, 1971). As procedural (instrumental) understanding is concerned with drawing on mathematical facts and carrying out algorithms (Mutawah et al., 2019), when such an algorithm is encountered, which does not fit into an existing schema, a lack of conceptual (relational) understanding may prove a challenge to the execution of this new algorithm. As Skemp (1971) argues, this may particularly be the case where a 'mathematical mismatch' (p.12) occurs, in which the pupil's goal is to understand instrumentally, however, is taught by a teacher who intends relational understanding. In contrast, he goes on to argue that the alternative 'mathematical mismatch', where the pupil's goal is to understand relationally, yet is taught by a teacher whose goal is to ensure instructional understanding, may indeed be far more damaging and have longer term implications for the student. For the justification behind this, we can consider Skemp's (1971), phrase, when referring to instrumental understanding: 'rules without reasons' (p.12). Thus, a teacher whose goal is instructional understanding, will not necessarily ensure the justification for the rule or approach and the conceptual understanding behind it, and consequently students may not have the capacity to draw on the conceptual knowledge to adapt such rules or approaches to unfamiliar contexts. In support of this, Mutawah et al. (2019) suggest that the reason for many children demonstrating poor performance in problem solving, particularly real-life problem solving, is a direct consequence of a lack of conceptual understanding, preventing them from making connections between the concepts and the problem-solving situation.

Nevertheless, there are arguments for incorporating both types of understanding within mathematical instruction. A secure instrumental understanding may provide pupils with the tools to devise a step-by-step plan to solve a mathematical word problem (Polya, 1945) and thus develop a more secure procedural fluency – knowing when, and how, to use these procedures, and applying them fluently and efficiently (Mutawah et al., 2019). Within this, pupils are required to draw upon their existing knowledge schemas to select an appropriate strategy for solving the problem. However, Skemp (1978) argues that whilst the short-term benefits of this may be apparent, the contexts to which this type of understanding are successful, may well be limited.

Relational understanding, on the other hand, enables pupils to develop a conceptual structure (or schema) from which an unlimited number of plans can be produced (Skemp, 1978), clearly demonstrating the long term benefits of such understanding. As relational understanding requires a secure understanding of not only the rules, but the connections between the concepts and rules, it naturally may take longer for pupils to learn, in line with schema development, discussed earlier (Kintsch & Greeno, 1985; Maglicco, 2016; Mayer, 1985; Peled & Wittrock, 1990; Schoenfeld, 1982). Although, Skemp (1978) argues that it is ultimately easier to remember these connections, despite taking longer to teach (p.24). The development of such connections may be facilitated through the use of representations (Barmby et al., 2013)(discussed in chapter 2.2.8). However, the long term benefits of adaptability to new tasks may be argued as a reason for ensuring relational understanding, in order to ensure secure and fluent mastery of mathematical concepts (DfE, 2013). Consequently, for those individuals displaying WCC, this may be pertinent to the pedagogical implications of the teaching of word problem-solving within the mathematics classroom.

When considering the above studies, it can be argued that both instrumental and relational understanding should underpin teaching and learning, however, as discussed, caution over the mismatch between the two should be applied. When teaching for relational understanding, consideration should be given as to whether pupils are actually learning through relational understanding, or simply attempting to draw upon the instrumental aspects and vice-versa (Mutawah et al., 2019; Skemp, 1971, 1978).

Nevertheless, relational understanding (which, although defined in the same way, is referred to as conceptual understanding in the current National Curriculum (DfE, 2013)), often requires much more teaching time (Skemp, 1978). Ensuring sufficient teaching time to enable pupils to develop their conceptual understanding and the volume of content within the curriculum may present tensions in the school context. Gningue et al. (2014), suggest that a demanding syllabus may prevent pupils moving from Bruner's (1985) iconic to symbolic representation stage (which requires the development of secure relational understanding) when learning mathematical concepts. In order to alleviate these tensions, careful consideration to pedagogical strategies used within mathematics must be considered, as it is suggested that investment in carefully considered instructional time needs to be considerable in order to yield success (Kilpatrick, 1985).

Schoenfeld (1982) argues that it is not sufficient to be able to be aware of, and to use a range of strategies, but more importantly, 'to know which to use and when' (Schoenfeld, 1980, p. 795), which is often missing in problem-solving instruction (Lester, 1985). Schoenfeld (1980) goes as far as to suggest that both the 'mastery of the basic [problem-solving] techniques', and the ability to make an 'appropriate choice of strategy' may indeed be 'necessary (and perhaps sufficient) factors for success in problem-solving' (Schoenfeld, 1982, p. 32). In addition to pupils' ability regarding the choice of strategy to utilise within mathematical problem solving, those pupils who are trained and equipped with problem solving skills, will be able to 'skilfully select important and relevant information, analyse it and draw conclusions' (Siregar et al., 2019, p. 756). In contrast, those pupils with a poor problem-solving skillset often fail to 'understand the true meaning of the problem' (ibid.).

Thus, when it comes to assessing overall problem-solving success of individuals, there may be a need to look further than simply a count of correct responses. Given the wide variety of factors already discussed in relation to problem solving success, in addition to correct response, measures which take into account metacognitive factors such as 'willingness to attempt difficult problems or perseverance in attempting problem solutions' may be key areas for consideration (Silver, 1985, p. 257). Coupled with these measures, it is also suggested that pupils' own perceptions of problem-solving competence may also contribute as a key affective component to problem-solving success (McLeod, 1985; Schoenfeld, 1985). Drawing on this work, the current study considers pupils' self-perception of mathematical ability as a potential condition in successful mathematical problem solving (discussed further in chapter 3.4.2).

The studies discussed within this section begin to highlight the complexities involved with problem solving along with the potential implications of working memory and cognitive load on problem-solving performance. Some evidence of the use of visual representations within problem solving have been discussed; this will be explored more later. However, one common type of mathematical problem, employed in primary schools, is the word problem, which is now discussed.

2.2.5 Word problems in mathematical problem solving

As mentioned in line with the example used earlier, according to Boonen et al. (2013), mathematical word problems may be defined as 'any mathematical exercise where significant background information on the problem is presented as text, rather than in mathematical notation' (Boonen et al., 2013, p. 271). The term mathematical word problems still covers a broad spectrum, as they may be 'algorithmic (where a set of rules needs to be followed to reach the solution), non-algorithmic (multiple strategies can be used), open (multiple solutions, usually requiring the search for patterns or relationships), or closed (a single solution)' (Chapman, 2006, p. 211). Furthermore, to reach a solution, word problems may require the problem solver to perform a single-step or multiple steps. However, despite the range of types of word problem in existence, what is common to them all, is the linguistic representation of a mathematical situation (Gerofsky, 1996; Greer, 1997).

It is suggested that mathematical word problems form a unique genre containing their own special characteristics (Kintsch & Greeno, 1985). Using the example discussed earlier, these can be explored:

- Reference to sets (Joe's set of marbles and Tom's set of marbles)
- Special presuppositions (all of the relevant information is given in the question. For example, Joe did not lose any of his marbles)
- Special comprehension strategies (the consideration of sharing the marbles or exchanging them with one another, as we may experience in real life, is dismissed)

(Kintsch & Greeno, 1985, p. 125).

In addition to these characteristics, according to Gerofsky (1996, p.37), most word problems consist of a 'three-component compositional structure':

- A 'set-up' component (e.g. characters, location)
- An information component (information needed to solve the problem)
- A question (the goal).

Within the example above, using Gerofsky's (1996) structure, Joe and Tom would be the set-up component; three marbles and five marbles would be the information component; and finally, how many […] altogether would constitute the question component.

Within such a question, a variety of skills are required to, and may influence the likelihood of, reaching the correct solution. In addition to pupils' arithmetic skill and algorithmic computation ability, which may appear to be obvious, it is suggested that the following may also be involved in reaching the solution:

- Long-term memory (the need to access mathematical facts)
- Alternative behaviours (the need to inhibit irrelevant stimuli)
- Non-verbal problem-solving skills (the ability to complete patterns presented visually)
- Language ability
- Reading comprehension
- Concept formation

(Fuchs et al., 2006, p. 31).

When considering the influence of reading comprehension, along with word decoding (discussed earlier), a number of studies have highlighted the significance of this as a predictor of word problem-solving ability (Bae et al., 2015; Björn et al., 2016; Boonen et al., 2013; Jones et al., 2009; Oswald et al., 2016; Özsoy, 2015; Wei et al., 2015; Whitby & Mancil, 2009), indicating the potential significance of this as a condition for analysis within the current study. Adding to this evidence on the significance of reading comprehension, is the model proposed by Kintsch & Greeno (1985) (discussed above), which focuses on the interaction between comprehension and word problem-solving ability. Within their model, it is suggested that the semantic structure of word problems is used to form schemas, within which sets of information are represented. In accordance with their model, each 'set schema' has four attributes:

- Object (e.g. a common noun)
- Quantity (the cardinality of the set)
- Specification (information which distinguishes it from other sets)
- Role (a relational term identifying the set's role in a higher-order level structure, which includes other sets)

(Kintsch & Greeno, 1985, p. 114)

Again, by considering the marble problem from earlier alongside this model, the word problem can be represented as the following sets:

Joe has three marbles.	Object: marbles
	Quantity: 3
	Specification: Joe
	Role: Subset (S1)
Tom has five marbles	Object: marbles
	Quantity: 5
	Specification: Tom
	Role: Subset (S2)
How many marbles altogether?	Object: marbles
	Quantity: 3 and 5 (goal)
	Specification: Joe and Tom
	Role: Superset (S3)

The goal of this word problem would thus involve counting all (marbles) to make a superset from the two subsets:

$$S3 = S1 + S2$$

(Kintsch & Greeno, 1985, p. 113).

Consequently, considering Kitsch and Greeno’s (1985) model, along with schema theory (Maglicco, 2016) and cognitive load theory (Sweller, 2011; Sweller et al., 2019), it can be reasonably argued that different word problems will generate different cognitive loads on the short-term buffer memory, and that the individual capacity of this may influence pupils’ abilities to reach the correct solution. This cognitive load may also be influenced by how the word problems are presented and the approach used to solve them. However, as Özsoy (2016) reminds us, reaching the correct solution for a mathematical word problem does not necessarily indicate possession of the relevant problem-solving skills. Such a correct solution may well have been reached without a secure understanding and reasoning of the strategy applied, thus something for educators to be perceptive of when teaching and assessing pupils’ word problem-solving skills. On the other hand, the high demand on the EF skills, posed by memory search and generation of a response (Desaunay et al., 2019), may be impaired in those individuals with diminished EF skills, therefore leading to incorrect responses, despite utilising appropriate problem solving skills.

2.2.6 Influences on individual’s problem-solving ability

Findings from previous studies discussed thus far (Bae et al., 2015; Björn et al., 2016; Desaunay et al., 2019; Fuchs et al., 2006; Kintsch & Greeno, 1985; Mayer, 1989; Schoenfeld, 2018; Utami & Warniasih, 2019; Wen, 2018), indicate that the following conditions are all likely to influence pupils’ overall ability to solve mathematical word problems:

- Reading comprehension ability;
- Cognitive load required within each set schema of the word problem;

- Individual’s short-term memory and working memory capacity;
- Verbal ability;
- Mathematical understanding – procedural and conceptual;
- Skills in the executive functions.

However, what remains unclear is the impact of combinations of these specific factors on the overall word problem-solving success of autistic individuals, which this study seeks to uncover.

Research into weaknesses in mathematical word-problem solving, particularly when focused on individuals with mathematical and/or learning disabilities, often divides the skill sets of focus into procedural skills (computational fluency and fact retrieval) and conceptual skills (number sense and problem solving skills) (Morin et al., 2017, p. 92). The two sets of procedural and conceptual skills, proposed by Morin et al. (2017), can be related to relational and instrumental understanding (discussed above). Fuchs et al. (2006) propose that an underlying assumption within much of the literature suggests a hierarchical development of these skills. This assumption implies that pupils’ skills in manipulation of numbers (procedural understanding), which also tends to be the focus of much traditional assessment in mathematics, precedes one’s understanding of numerical relations (conceptual understanding), which then enables successful word problem solving (Fuchs et al., 2006; Rosli et al., 2013).

Morin et al. (2017) suggest that ineffective word-problem solvers may have difficulties understanding the mathematical concepts embedded within word problems, calculation difficulties, and do not use problem solving strategies effectively. These findings corroborate a previous study, indicating that numerous studies suggest a multitude of causes for difficulties in problem solving, including reading ability, cognition, metacognition, working memory, language ability, visuo-spatial skills, low intelligence and motivation (Maglicco, 2016).

Drawing further on the work by Morin et al. (2017), the following diagram can be used to represent the potential steps involved in solving mathematical word problems:

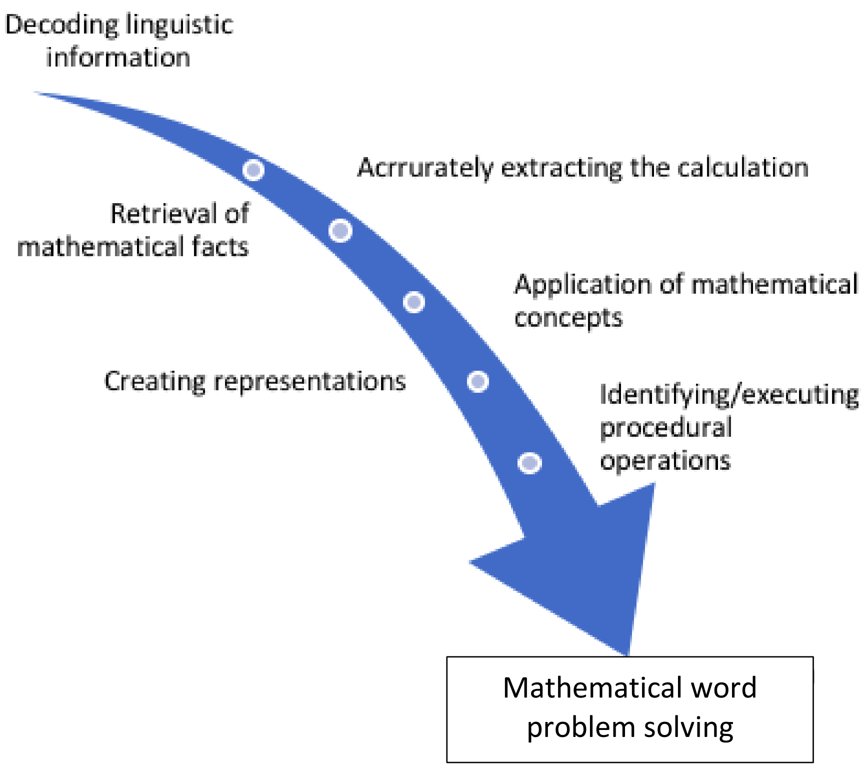


Figure 3: The multifaceted tasks involved in mathematical problem solving (Based on Morin et al., 2017, p. 92)

When it comes to solving mathematical word problems, it is suggested that two skills are particularly supportive: the production of visuo-schematic representations, which can support the visuo-spatial working memory of students with mathematical difficulties (Swanson et al., 2013); and relational processing that drives the correct relation between the solution-relevant elements from the text (Boonen et al., 2013). However, the role of linguistic or symbolic representation and interpretation of word problems are not necessarily considered by the authors in this study, thus may not be applicable to all mathematical problem solving.

The following section continues the theme of the influence of the classroom practices and pedagogy, coupled with the focus on the importance of the interplay of linguistics and comprehension in word problem solving ability. Let us now turn to consider a specific type of word problem in mathematics: the 'real-life' word problem, which often present their own unique challenges due to the concept of 'reality'. Furthermore, 'real-life' word problems are often used to exemplify the application of the bar model approach, thus will be the focus of this study.

2.2.7 The 'reality' of real-life word problems in mathematics

Within this section, the importance of language use, context and types of knowledge expected to be drawn upon by pupils attempting to solve such problems is discussed. Schoenfeld's (1991) notion of 'suspension of sense making' (p.311) is explored through some key theories and research around the types of knowledge required to understand and interpret 'real-life' word problems within the context of the mathematics classroom. Through some specific, worked examples, the unique nature of 'real-life' word problems is uncovered, leading us to question the 'reality' of such problems.

The 'real-life' word problem can be conceptualised as 'a written description of a [real-life] situation' (Greer, 1997, p. 300), and in order to reach the correct solution, pupils must make 'reasonable assumptions and construct a model for mathematics, not reality' (p.300). According to Cooper & Harries (2002), when attempting to solve a 'real-life' word problem in the mathematics classroom, pupils have a tendency to process the reality of the situation through inhibiting what is 'realistically meaningful' beyond the classroom (p.3).

Let us use an illustration to exemplify this point:

The flask is filled from a tap at a constant rate. If the depth of the water is 2.4cm after 10 seconds, about how deep will the water be after 30 seconds?

(Greer, 1997, p. 294)

Within the mathematics classroom, the expected solution to such a question would be 7.2cm (2.4cm x 3). However, as the flask is conical in the image, the 'reality' of this would certainly result in a very different solution. Various studies have used this particular question to ascertain how many pupils actually consider the 'realistic constraints' (Greer, 1997, p. 294) of such a question. The findings from across the studies suggested a high level of consistency:

Study	**% of students taking account of any realistic constraints**
13-14-year olds in Northern Ireland (Greer, 1993)	6%
10-11-year olds in Belgium (Verschaffel et al., 1994)	3%
10-12-year olds in Switzerland (Staub & Reusser, 1995)	0%

Table 2: Pupils' consideration of realistic constraints within mathematical problem solving (Greer, 1997)

As Greer (1997) highlights, findings from these studies demonstrate how pupils often fail to take account of the 'real-world situation described within the problem' (p.294). Such studies, and the claim made by Greer (1997) support Schoenfeld's (1991) argument that the culture of mathematics within the classroom promotes the 'suspension of sense-making' (p.311). In addition to this is the influence of the social demands of the classroom, where students are expected to grasp the knowledge and procedures presented to them via their teachers or through textbooks (Gravemeijer, 2020). There is an expectation that a solution is found, thus, the requirement for a numerical answer to such a question may indeed lead pupils to submit an answer that does not necessarily align with the image presented.

When faced with such problems, pupils must not only draw on their semantic and comprehension knowledge of the text, but also on their ability to discriminate between their experiential knowledge, based on the reality of contexts, and their knowledge of the culture of classroom mathematics, which must be based on reasonable assumptions (Gravemeijer, 2020; Greer, 1997; Van den Heuvel-Panhuizen, 2020; von Glaserfeld, 1987). One explanation for such assumptions required to solve 'real-life' mathematics within the mathematics classroom, through the 'suspension of sense-making' (Schoenfeld, 1991, p. 311), is the influence of the 'didactical contract' (Deliyianni et al., 2009, p. 95; Greer, 1997), where pupils become 'enculturated in the world of school mathematics' (Gerofsky, 1996, p. 38). With classrooms being diverse learning communities, it should also be taken into consideration that pupils from different cultural backgrounds may also understand and interpret word problems in different ways (Ngeno et al., 2019).

A 'didactical contract' refers to 'the implicit and explicit set of rules that dictate the interactions between the pupil, the teacher and mathematical knowledge used within the classroom' (Deliyianni et al., 2009, p. 95; Greer, 1997). Deliyianni et al. (2009) suggest that pupils' cognitive processes used for mathematical problem solving, often follow these rules, which in turn, can lead to 'dysfunction and deficiencies' in mathematics learning (p.97). According to this theory, which, as educators we should perhaps be challenging, the nature of many mathematical word problems and accompanying classroom teaching does not adequately enable pupils to draw on, and utilise, their everyday knowledge of the real world. As a result, pupils have a 'tendency to focus on the syntactic and linguistic structures of mathematics' and follow a simplistic model of problem solving representation, often resulting in failure (Deliyianni et al., 2009, p. 99). They conclude their study by recommending that any breach of this 'didactical contract', such as through the use and understanding of effective visual representations (discussed later), may indeed enhance the problem-solving success of pupils, by encouraging them to search for deeper meanings within the problem and drawing on their real-world knowledge to solve and represent the solution successfully. Consequently, it is suggested that 'The challenge for teachers is to provide students with problems that draw on their experience of reality, rather than asking them to suspend it. Realistic does not mean that problems necessarily involve real contexts, but rather they make students think in "real" ways' (Klerlein & Hervey, 2019, p. 5).

The language used in real-life word problems is often not the same language as children would use, or encounter, in their everyday lives (Ngeno et al., 2019), thus compounding this issue even further. However, it is suggested that pupils may be more motivated by mathematical tasks, which are contextualised within real problems, rather than tasks using symbols alone and the cognitive demands of contextualised tasks are different. Consequently, contextualised tasks, which draw on knowledge beyond the classroom, may provide support to students and increase the accessibility of the task to assist pupils with their understanding of mathematical rules within the problem (Van den Heuvel-Panhuizen, 2020).

Drawing on the influence of experiential knowledge and didactical contracts, Greer (1997) established the following schematic diagram to represent the interplay of such factors within the real-life word problem-solving process:

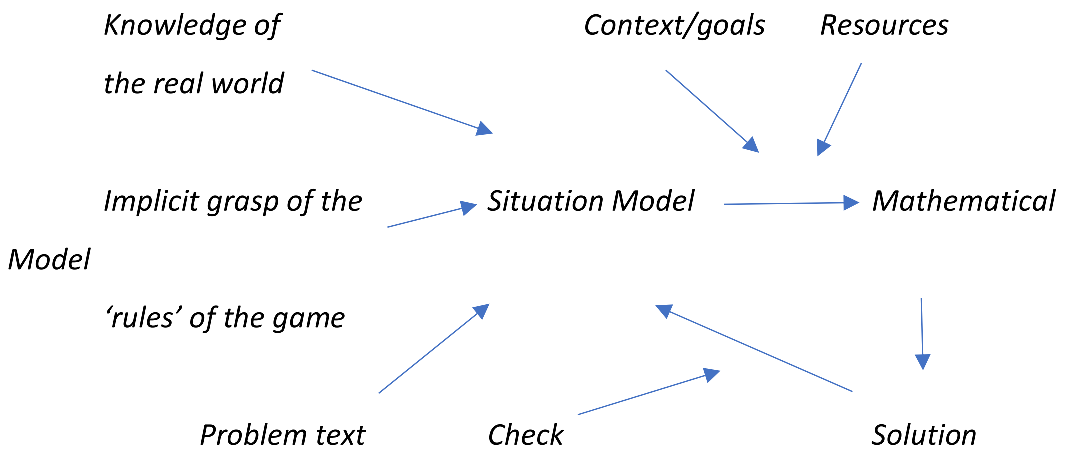


Figure 4: Schematic diagram of the factors influencing the modelling of real-life word problems (Greer, 1997, p. 301)

When considering the pedagogical implications of teaching 'real-life' word problems in the mathematics classroom, we can draw on Bruner's (1966; 1985; 1986) two modes of knowing: paradigmatic and narrative. Paradigmatic knowing is a logico-scientific concept, in which pupils disregard any particular social context associated with the mathematical structures or models. In contrast, narrative knowing is humanistic in nature and considers the social context of the problem (Chapman, 2006). As 'real-life' word problems 'contain both logico-scientific and humanistic components' (Chapman, 2006, p. 227), teachers should consider both approaches within their instruction, as both models are 'amenable to complementary use' (Bruner, 1986, p. 97). Through discussing and questioning such 'plausible assumptions' (Greer, 1997, p. 297) with pupils, consideration as to how closely the mathematical modelling aligns with the reality of the context of the question can be explored (Davis & Hersh, 1981; Greer, 1997; Staub & Reusser, 1995). Although contextualisation of a task may provide meaning to the numbers, it does not necessarily make the task useful – educators should consider whether the question would normally be asked within the given context, or whether the task is simply 'dressed up' in order to hide the specific mathematical task (Van den Heuvel-Panhuizen, 2020, p. 37). Two examples are provided to exemplify this concept:

> *How many quarters of an hour go into three and a half hours?*
>
> *A doctor […] has consultations in the morning from 0830 to 1200. The patients have a consultation visit of a quarter of an hour. How many patients can the doctor see?*

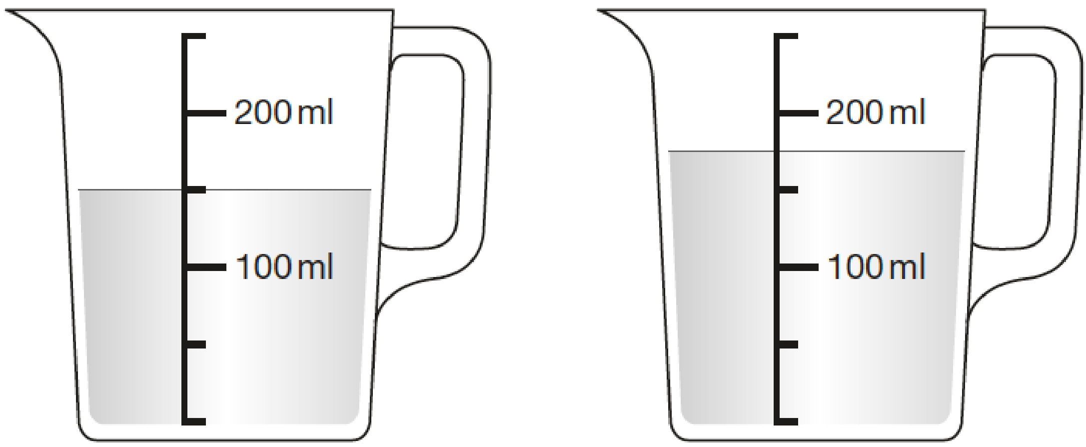
(Heuval-Panhuizen, 2020, pp.37-8)

In the example above, the first question is a 'dressed up' question – containing no real context or justification for the question. However, the second question, which requires pupils to carry out the same calculation, is contextualised within a real-life context and provides a justification for the mathematical calculation required. Furthermore, the context is likely to be imaginable, if not experientially real, for the students (Van den Heuvel-Panhuizen, 2020). Such discussions may promote the breach of any didactical contracts established within the classroom, to support the development of pupils' word problem-solving ability.

The significance of such pedagogical approaches is exemplified by Cooper and Harries (2002), who consider ‘real-life’ word problems within the context of high-stakes testing, adding further to the justification of the current study. Should the flask example, as discussed earlier, appear on a mathematics test, pupils may in fact be ‘penalised’ for considering the ‘realistic considerations’ of the question in reaching their solution (p.1). A more recent example in a Key Stage 2 SATs paper (DfE, 2018) exemplifies this problem further. Like the conical flask example, there is an expectation that the pupils will not consider the tapered shape of the jug in relation to the equally spaced measure line used within the diagram, when interpreting the problem and calculating the expected solution (275ml).

Stefan has **600 millilitres** of water in a bottle.

He pours some of the water into two measuring jugs as shown.



How many millilitres of water are left in Stefan’s bottle?

Figure 5: An example of the need to suspend reality when solving mathematical word problems (DfE, 2018, sec. 11)

The examples and studies referred to within this section suggest that ‘real-life’ word problems ‘have their own rules’ (Greer, 1997, p. 304) and that these rules become implicit through pupils’ enculturation and socialisation into the classroom culture

(Gerofsky, 1996; Wyndhamn & Säljö, 1997). These rules build on the 'special characteristics' of word problems discussed earlier (Kintsch & Greeno, 1985, p. 125).

2.2.8 Visual representations in mathematical problem solving

This section begins by exploring the concept of representations before moving on to understand different 'types' of representations within mathematics. Following on from this, the role of visual representations in mathematical word problems are then considered. Evidence is provided to suggest links between the use of visual representations within mathematics and the impact on pupils' cognitive load (discussed earlier). A number of studies are drawn upon, exploring the impact of visual representations in mathematical problem-solving success with different sub-groups of learners (Barmby et al., 2013; Cooper et al., 2018; Kalyuga, 2013; Kalyuga et al., 1998). Finally, the bar model, as a specific type of visual schematic representation, on which this research is based, is explored in more detail, drawing on key problem-solving theories discussed in the previous sections.

Visual representations may be external or internal. External representations are those diagrams or schematics, which are presented, or represented, to support or model pupils' mathematical understanding and to illustrate mathematical concepts (Bolden et al., 2015). In contrast, internal representations are mentally produced, thus relying on the recall or visual imagery of a schematic representation from pupils' memory, or the 'mental images corresponding to internal formulations we construct of reality' (Dufour-Janvier et al., 1987, p. 109). From an epistemological perspective, such internal representations are shaped by our own experiences of the world.

External visualisations may include mathematical diagrams, graphs or charts to support the presentation and processing of mathematical and contextual information (visual-schematic), or may simply be decorative images, intended to capture students' interest (pictorial). Such visual-schematic representations support the construction of a coherent visualisation, which embodies the relevant textual information required for the solution. Furthermore, they are an important part of mathematical problem solving and can support students' connections between mathematical knowledge (Bolden et al., 2015; Hegarty & Kozhevnikov, 1999). Although the current study is less interested in pictorial visualisations, it is worth noting that both types may impact upon students' working memory and cognitive capacity to process information, be it in a supportive or detrimental manner.

The following word problems provide an example of each of these types of external visual representations within mathematics:

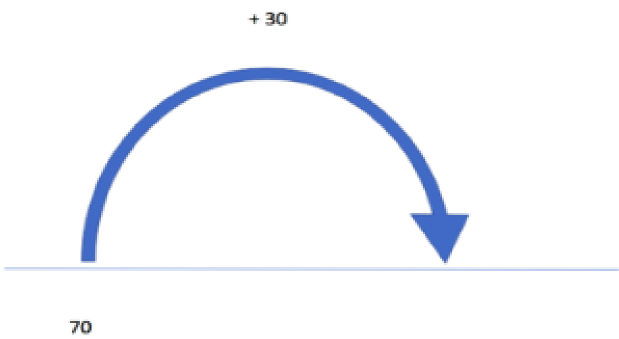
a. An example of a pictorial representation

On a school trip were 70 boys and 30 girls. How many went on the school trip altogether?

Here, the visual representation, is used more for illustrative purposes and whilst providing a prompt, or context, for the pupil, it provides little support in terms of the mathematical requirements of the question.

b. An example of a schematic representation:

On a school trip were 70 boys and 30 girls. How many went on the school trip altogether?



In this example, the schematic representation enables the relationships between the quantities in the word problem to be visualised, thus potentially reducing the cognitive demands of the question.

Schematic representations require students to ‘encode the spatial relations described within the problem’, whereas pictorial representations require students to ‘encode the visual appearance of the objects described in the problem’ (Deliyianni et al., 2009, p. 98). When investigating these two types of representations, Hegarty and Kozhevnikov (1999), propose that only schematic representations are positively correlated with problem solving success, whereas, in contrast, pictorial representations (such as the conical flask example discussed earlier) are negatively correlated. However, there is little evidence to suggest a causal relationship from this study, consequently consideration must be given to the potential of other intervening factors, discussed within this literature review.

Adding to the complexity of the concept of representations, ‘a representation does not represent itself – it needs interpreting and, to be interpreted, it needs an interpreter’ (von Glaserfeld, 1987, p. 216). In order for this to happen, teachers must facilitate students’ learning to interpret the representations to ensure they match those interpretations of the teacher (Barmby et al., 2013). Therefore, the use of visualisations requires either translations from an external representation to an internal understanding (e.g. interpreting a graph) or a translation from an internal model to form an external representation of the pupil’s understanding (Gierus, 2011). Such translation of representations may result in a ‘learning paradox’ (Gravemeijer, 2020, p. 221). Here, it is argued that where models and representations are used, learning is considered a process in which pupils construct a mental representation mirroring the mathematical features of an external representation. If this is the case, then any interpretation of the external representation is dependent on the pupils’ knowledge and understanding of the representation to be interpreted (ibid.). It is therefore important to note that when using external representations, there is a need to consider just how well pupils’ ‘cognitive structures match what they are intended to represent' (von Glaserfeld, 1987, pp. 4–5) and consequently, the 'linguistic reality of the constructs associated with' external representations (Kaput, 1987, p. 22).

When situated within the context of problem solving, the potential translations required between different modes of representation, can be summarised using the model below:

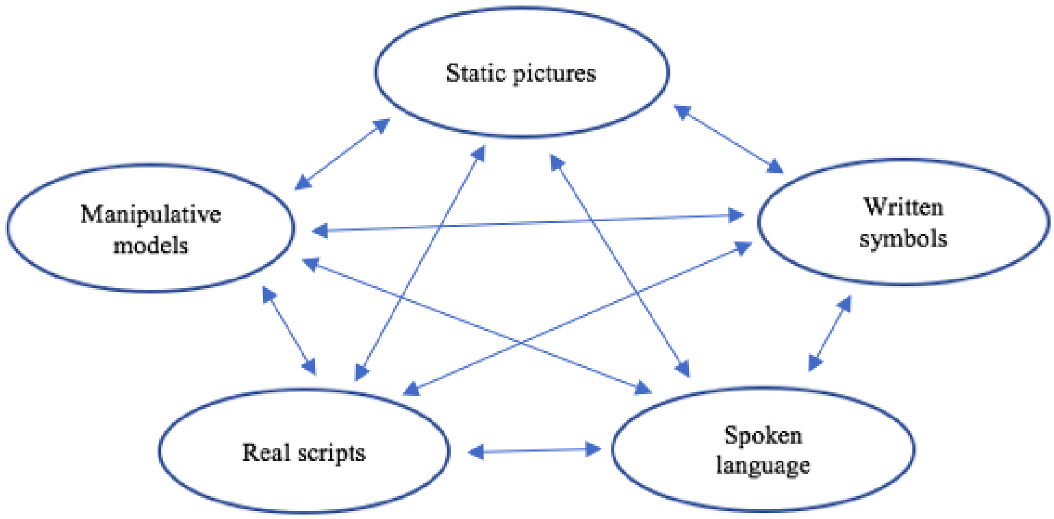


Figure 6: Translation pathways between different modes of representation within mathematical problem solving (Lesh et al., 1987, p. 34)

Crucially, according to Lesh et al. (1987), any difficulties associated with making such translations between different modes of representation, may have 'a significant influence on mathematical learning and problem-solving success' (p.36). Nevertheless, supporting students to overcome these difficulties and considering a dynamic approach to learning through models and representations, may facilitate problem-solving success (Gravemeijer, 2020; Lesh et al., 1987).

It is suggested that such 'learning paradoxes' (Gravemeijer, 2020, p. 221) may disband when the learning process is considered to be more dynamic and pupils' use of models (and mathematical symbols) are developed in a 'bottom-up' manner (ibid. p.221). This can be exemplified through the success of the empty number line as a mathematical model. The empty number line, which is one commonly used example of a visual representation that is found to enhance students' understanding and development of mathematical concepts, is– 'a visual representation that illustrates the order and magnitude of numbers' (Woods et al., 2017, p. 230). Use of the number line, according to Booth & Siegler (2008), is commonly observed in the primary mathematics classroom and further research suggests that the use of such visual representation [the empty number line] can support students' development of number sense (Booth & Siegler, 2008). The development of such understanding may underpin more complex mathematical concepts such as place value and comparison, by helping students to create a mental representation of the order and magnitude of numbers (Woods et al., 2017). Although initially used in calculations closely matched to the situations in contextual problems, further engagement with the empty number line is suggested to support pupils' understanding of number relations and pupils' mental computation strategies (Bobis & Bobis, 2005; Gravemeijer, 2020). Consequently, the empty number line (or any other mathematical model) may develop pupils' formal mathematical reasoning, rather than simply a way of representing a contextual problem. Therefore, as students become more experienced with tackling similar problems, attention is directed towards the mathematical relations involved and structures (Lesh et al., 1987), rather than the problem context itself. This process is referred to as the 'emergent-modelling design heuristic' (Gravemeijer, 2020, p. 223) and, as such, use of the bar model, as a specific representation in mathematical problem solving, may well follow suit, similar to the success of the empty number line.

Siregar et al. (2019) suggest that in terms of improving pupils' problem-solving skills in mathematics, one approach that teachers should use is to select a model (or representation) which is 'precise and nuanced to the competency of the students' (p.758). Furthermore, and potentially linked to skills in EF, Lesh et al. (1987) argue that the most proficient problem solvers tend to demonstrate flexibility in their 'use of a variety of relevant representational systems' (p.38). However, whichever mathematical model is used, consideration must be given to the level of understanding reached by the pupils. The use of such visual representations can support students to reason and develop connections between their experiences and the mathematical concepts. Nevertheless, it is important to remember that such representations, are merely the 'tools' for facilitating conceptual understanding in mathematics, rather than the concept itself (Gravemeijer, 2020; Woods et al., 2017, p. 231).

Four levels of activity are suggested by Gravemeijer (2020), where the model is used to represent what is experientially real for students (the situated level) to a point where students begin to reason about the mathematical relationships represented by the model (the general level) (p.226). At this level (and the final level where formal mathematical reasoning is developed) the model begins to lose its dependence on context-specific image representation and moves towards becoming a model of mathematical relations (Gravemeijer, 2020). Nevertheless, perhaps even more important, is that these proficient problem-solvers can 'instinctively switch to the most convenient representation at any point in the [problem-solving] process' (Lesh et al., 1987, p. 38).

As von Glaserfeld (1987) reminds us, it is important to remember that for pupils to construct representations, they 'must acquire the ability to imagine or visualise' through drawing on their experiential knowledge (p.7). Visual representations may cause confusion to some students because of the dual role they play - students have to learn about visual representations, (how they depict information), but also how to extract new information from the visual representations (interpret them) (Deliyianni et al., 2009; Rau, 2017). Learning about visual representations requires students to acquire 'representational competencies' (knowledge and skills that enable them to use visual representations to reason and solve tasks). However, the acquisition of representational competencies is cognitively demanding and requires instructional support (Rau, 2017, p. 719).

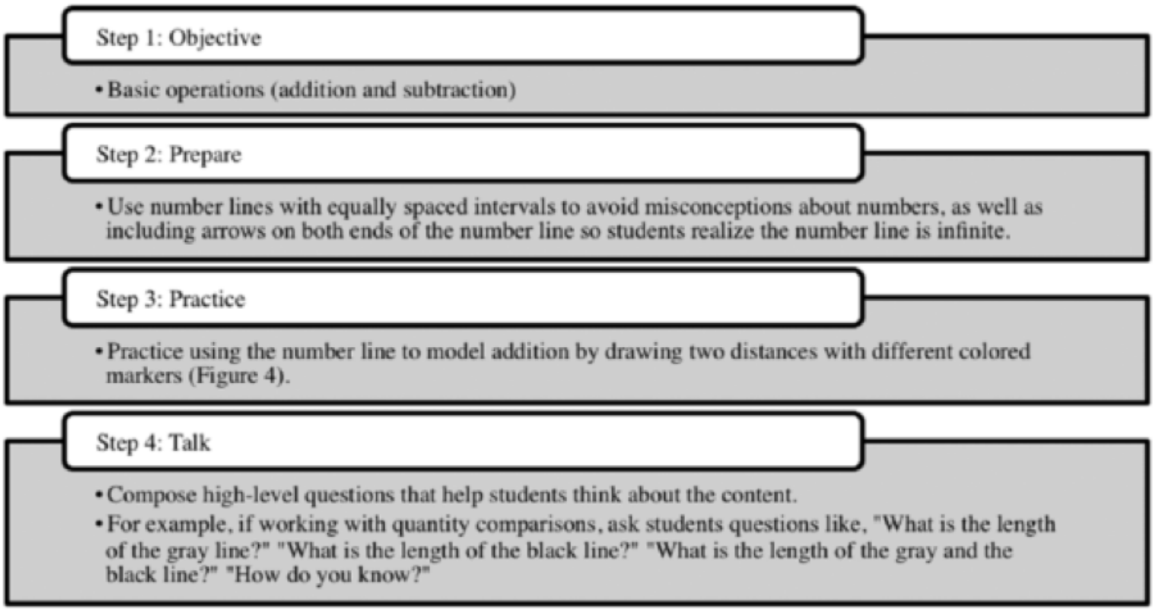


Figure 7: Systematic instructional steps for incorporating the number line into basic operations (Woods et al., 2017, p. 232)

When considering the first role of representations, as described by Rau (2017), Woods (2017) offers a suggestion of the steps involved in the incorporation of systematic instruction of the number line into classroom practice (figure 7), which clearly demonstrates how such a 'tool' must be embedded into mathematics instruction, whilst also supporting the development of more complex mathematical content understanding. The benefits of visual representations to support mathematics learning through understanding such representations as a 'tool' are summarised by Woods et al. (2017), who claim that 'systematically incorporating precise visual representations [like the number line] into mathematics instruction […] will not only support the development of students' understanding of increasingly complex mathematics concepts […] but also equip students with a mathematical tool that can support their thinking over time' (p. 235).

The use of systematic instruction to enhance learning, when combined with visual representations, is apparent in an earlier study, by Shin et al. (2015), who found that ten out of seventeen studies, in their synthesised intervention study on the improvement of fraction skills, combined both of these approaches to yield positive outcomes on student performance (Shin & Bryant, 2015). Although most of these studies (15) focused on secondary school pupils, and the focus was specifically on the development of conceptual understanding in the field of fractions, the study nevertheless highlights the potential power of visual representations in mathematics learning, when coupled with systematic instruction, as an approach for developing pupils' mathematical understanding.

Within mathematics education, it is commonplace to draw upon, and utilise multiple visual representations to support pupils' mathematical understanding. In terms of classroom application of the use of multiple external representations, Ainsworth (2006) defined three critical functions (those representations used within learning activities (Rau, 2017)):

1. Complementary processes, where different representations can provide different types of information and facilitate different types of learning process;
2. Constraining interpretation, where representations, which are familiar to the student, can be employed to support the interpretation of other representations;
3. Construction of deeper learning, where information from different representations can be integrated.

These functions can be demonstrated through the following example, which makes use of two visual external representations – an array and an empty number line:

(3 x 4) + (2 x 7) =

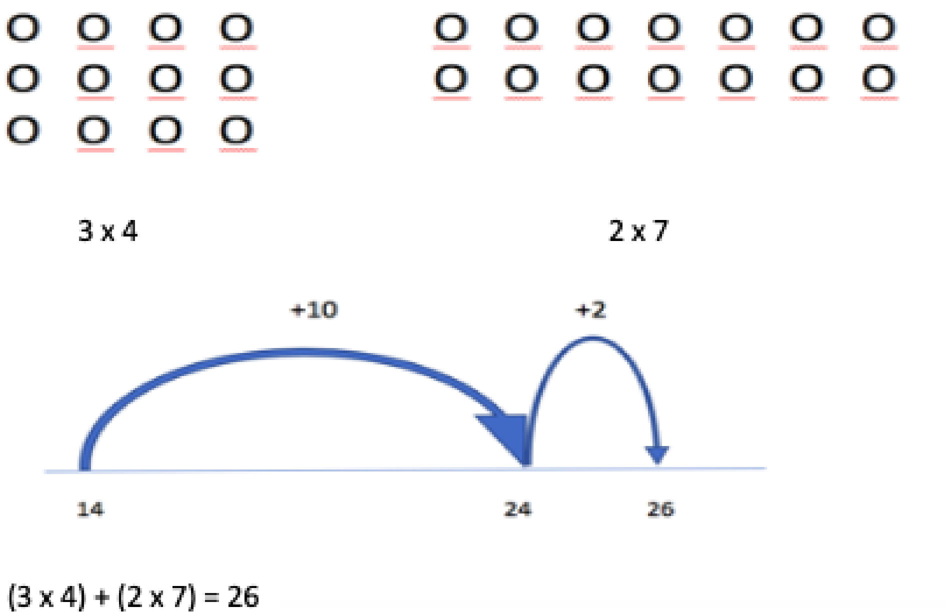


Figure 8: The use of multiple external representations to model the calculation (3 x 4) + (2 x 7) = 26

In the example above, the array provides information for calculating the multiplication aspects of the calculation, whilst the empty number line supports the addition aspect (complementary processes). The array can explain the commutative law of multiplication (Barmby et al., 2013), and where an array is familiar to the pupil, this can support the understanding of the derivation of the 'jumps' on the number line (constraining information). Finally, by integrating the information presented both in the array and the empty number line, connections between the two can be made (construction of deeper learning).

The three functions described by Ainsworth (2006), along with the cognitively demanding task of acquiring representational competencies (Rau, 2017), are all underpinned by cognitive load theory (Sweller, 2011; Sweller et al., 2019), which 'provides a solid framework for examining the effectiveness of visual representations and how they can support learners' cognitive processing' (Hsin & Paas, 2016, p. 71). Through employing different learning processes, drawing on prior knowledge of familiar representations, and integrating information from a range of domains, the cognitive load experienced by the student, during a problem-solving task, may be reduced. This supports the theory that visual representations can be used to reduce the cognitive load on students, thus enabling them to redirect their resources to processes that can enhance the understanding of the problem (Kalyuga, 2013). Ainsworth's (2006) idea of the construction of deeper learning, and Kalyuga's (2013) theory that visual representations can reduce students' cognitive load is supported by Hsin & Paas's (2016) study, carried out on fourth grade students in Taiwan (N=46). In their study it is claimed that the use of visual representations within 2-step problem-solving tasks led to an increase in learning performance and a reduction in cognitive load (Hsin & Paas, 2016). Whilst the findings from this study align with similar theories relating to visual representations reducing students' cognitive load (Ainsworth, 2006; Cooper et al., 2018; Kalyuga, 2013), the wide distribution around the mean scores with both the learning performance and perceived difficulty, suggest the difference between the two groups is not necessarily notable. In addition, the measure of perceived difficulty (interpreted as a measurement of cognitive load) is reliant upon students' accurately and truly scoring the task based on its level of difficulty. This measure is open to a significant degree of fallibility, as the students may not have wished to admit to higher levels of difficulty (therefore recording lower scores), or, as the mean scores are low in both groups, this may simply indicate a relatively simple set of problem-solving questions.

In addition to the potential influence of cognitive load theory (Sweller, 2011; Sweller et al., 2019), the use of, and explanation for, visual representations within mathematics can also be underpinned by the cognitive theory of multimedia learning (Mayer, 2005). Common to both of these theories is the notion that 'the structure of the cognitive system imposes limits that influence how learners select, organise and integrate information' (Cooper et al., 2018, p. 25). However, the cognitive theory of multimedia learning suggests that through combining words and images (or representations), the processing of information is shared between both visual and verbal channels (Rau, 2017). Multimedia learning theory suggests that 'students' learning from visual representations hinges on their ability to form accurate internal representations of the representations' referents and on their ability to integrate internal representations into a coherent mental model of the content' (Rau, 2017, p.

721). This theory thus implies the requirement of ‘representational competencies’, of which two types exist: conceptual and perceptual.

Conceptual representational competencies involve the ‘knowledge and skills students use to map visual representations to concepts, to make inferences based on visual representations, and to choose a particular visual representation for a task because it depicts relevant concepts’. In contrast, ‘perceptual representational competencies involve the ability to effortlessly and efficiently see meaning in visual representations’ (Rau, 2017, p. 722). Perceptual competencies can be described as ‘fluency’ in processing information from visual representations and relies upon experience-based learning (Kellman & Massey, 2013), requiring them to become confident in classifying and categorising tasks accordingly to the visual representations (Rau, 2017).

Although previous research into the impact of visual representations on mathematical problem solving ability are mixed (Ainsworth, 2006; Berends & van Lieshout, 2009; Cooper et al., 2018; Hegarty & Kozhevnikov, 1999; Hsin & Paas, 2016; Kalyuga, 2013; Magner et al., 2014), it appears that the findings often differ according to different sub-groups of students. This may be particularly the case with respect to students’ mathematical ability and interest in mathematics, and the effects of visual representations may be mediated by factors such as levels of prior knowledge, cognitive resources and demands on working memory (Cooper et al., 2018; Rau, 2017). It is suggested that those students with less prior mathematical knowledge often benefit more from detailed mathematical diagrams and representations, whereas those students with greater prior knowledge may benefit more from more sparse diagrams and representations due to a reduction in demands on working memory and cognitive load (Cooper et al., 2018; Kalyuga, 2007; Kalyuga et al., 1998). Furthermore, it is suggested that pupils’ competence in creating visual representations is related to their spatial ability and studies show that those pupils whose spatial ability is high, but whose verbal ability is low, tend to utilise their spatial skills when it comes to mathematical problem solving (Deliyianni et al., 2009). It is suggested that benefit may be gained from the visual storage of information through the use of representations for those students with learning difficulties (Hord et al., 2016, p. 15), in terms of supporting working memory capacity. When considering a different subgroup of learners, Cooper et al.'s (2018) study suggests that the use of illustrations (rather than mathematical diagrams) supported the problem solving accuracy of those students with higher mathematical ability and more positive attitudes towards mathematics, compared to those with lower mathematical ability and attitudes (Cooper et al., 2018). However, it may be argued that those pupils with higher mathematical ability and attitudes are less reliant on working memory and retrieval of information than those with lower abilities (Hord et al., 2016), thus enabling more cognitive capacity to be applied to solving the problem as well as processing the illustrations. Those students with lower mathematical ability or interest may be directing more cognitive resource into processing the illustrations, or simply becoming distracted by the illustrations, when this resource is a greater requirement for these students to process and solve and the mathematical problem.

When it comes to considering autistic pupils, this dual channel processing relies upon substantial integration and selection of information by the student, which may be hindered by difficulties with central coherence and skills in executive functions (discussed earlier). Through overloading the visual or verbal pathways, the cognitive resources available for solving the mathematical problem may be limited (Cooper et al., 2018).

In support of this, Boonen et al. (2013) suggest that the construction of visual schematic representations may be influenced by pupils' spatial ability, with which pupils with high-functioning autism (note the subgroup of autism here, discussed earlier) often present difficulties (Aagten-Murphy et al., 2015, p. 1). Boonen et al. (2013) go on to suggest that good word problem solvers often construct visual representations to facilitate their understanding, however the nature of these representations can determine their effectiveness. Finally, they suggest that the ability to construct visual schematic representations of word problems may be a necessary, but not always a sufficient condition, when it comes to solving mathematical word problems, which it is anticipated that qualitative comparative analysis (QCA) used within this study may substantiate or not.

A meta-analysis of studies focusing on students with learning difficulties, indicated that the use of diagrammatic representations during problem solving proved to benefit the students significantly (Barmby et al., 2013). In terms of the potential influence of WCC on constructing and manipulating visual representations, Lesh et al. (1987) concluded from their study that when students' attention focused on the representation as a whole, previously noticed details were overlooked. In contrast, when focusing on specific elements of the representation, students 'temporarily lost cognizance of others' (p.40). Although the participants within this study were NT children, this finding highlights the potential exacerbation of such behaviours when WCC is considered as an additional factor.

In contrast to Ainsworth's (2006) arguments for using multiple external representations to facilitate students' learning, previous research casts doubt over such use. It is suggested that the use of multiple representations can lead to 'syntactical rules of correspondence' between different representations, 'rather than constructing the concept itself via its representation' (Dufour-Janvier et al., 1987, p. 113), therefore supporting students to make connections between multiple representations (Barmby et al., 2013). Consequently, the current study focuses on just one type of representation – the bar model. Nevertheless, despite the findings from these previous studies, subsequent research has highlighted the need for further investigation into how visual representations can be incorporated into classroom practices within mathematics education (David & Tomaz, 2018).

Therefore, as can be seen, the range of visual representations used within mathematics is vast, as is their intended purpose, which may be to support conceptual or procedural understanding, or to simply attempt to add interest to the task. However, in support of Bruner's (1966) stages of mathematical representation, discussed earlier in the chapter, 'visual representations can help students assign meanings to the mathematics concepts they are learning, before they learn to use formal notation and work with abstract ideas' (Murata & Stewart, 2017, p. 406). Hence, visual representations may be used to bridge the gap between the enactive (concrete) stage of representation and the symbolic (abstract) stage, through the iconic (pictorial) stage.

The studies discussed thus far present some evidence for the benefits of using visual (schematic) representations within mathematics, particularly in mathematical problem solving. Whilst some specific representations (the number line) can be seen to reduce the overall cognitive load on some students' processing, the study by Cooper et al. (2018) has highlighted the potential sub-group differences in terms of the overall usefulness of visual representations and associates other mediating factors, such as prior mathematical knowledge, with the outcome. What is less clear at this stage, is the effectiveness of the bar model, as a specific schematic representation, and in addition, the impact of those mediating factors associated with autistic individuals, coupled with the use of such schematic representation, on overall mathematical problem-solving success for this population.

2.2.9 The bar model as a visual representation in mathematical problem solving

This section now explores the bar model, as a specific type of visual-schematic representation, in further detail. A brief introduction to the development of the mathematics curriculum in Singapore provides the rationale behind the adoption of the bar model as a tool to support mathematical problem solving. Following on from this, the theoretical frameworks underpinning the bar model are then explored, making links back to some of the key problem-solving theories discussed in previous sections.

2.2.9a Development of the bar model within the Singapore mathematics curriculum

The problem-solving framework underpinning the Singapore mathematics curriculum, places emphasis on mathematical processes and cognitive aspects of learning, in addition to the mathematical content of the curriculum, as can be seen in figure 9 below (Singapore Ministry of Education, 2012).

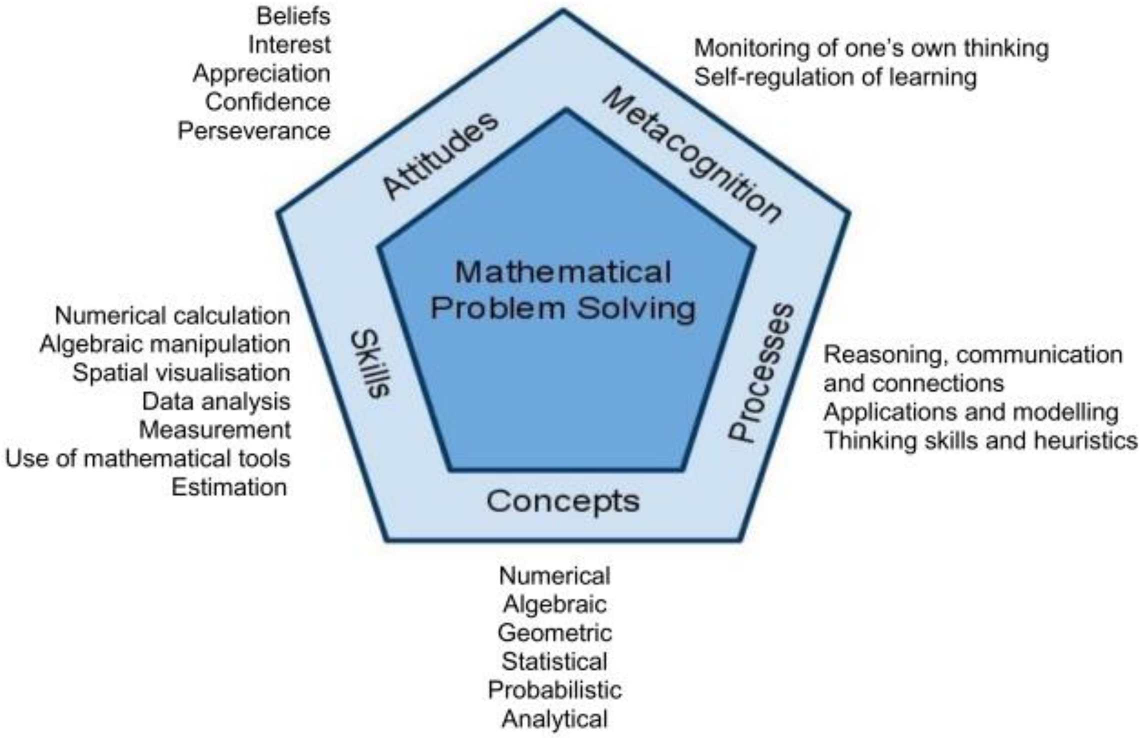


Figure 9: The mathematical problem-solving framework underpinning the Singapore Mathematics syllabus (Singapore Ministry of Education, 2012, p. 14)

The core aspect of problem solving as a goal and purpose, central to the mathematics curriculum in Singapore (figure 9), is distinctly different from the mathematics curriculum in the U.K., U.S.A., and Australia. In these latter three countries, although problem solving features heavily within the curriculum, it is considered as one aspect mathematics education, rather than the overall goal and purpose (Bingolbali & Bingolbali, 2018; DfE, 2013).

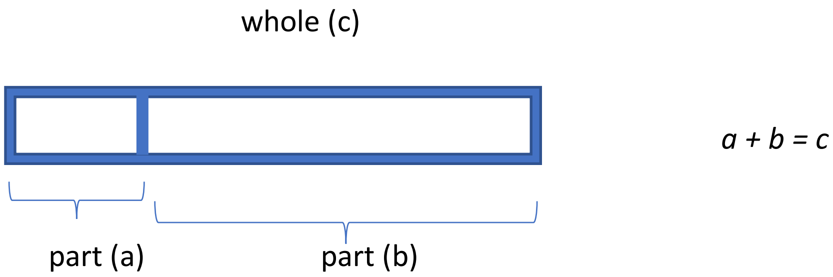
Unlike many Western curricula, the mathematics syllabus in Singapore focuses on depth, rather than breadth, and introduces topics and concepts at a slower pace, with a focus on problem solving skills, giving time for deep, conceptual understanding of the foundations of mathematics to be embedded, acting as building blocks for future, more complex concepts. Within problem solving, the focus is less directed on reaching the correct solution, but instead is more focused on exploring a variety of ways in which to tackle the problem. In support of the research into the lack of real-life application into mathematical problem-solving carried out by Deliyianni et al. (2009), the Singapore mathematics syllabus places a great emphasis on teachers providing realistic contexts for the application of mathematical skills and concepts (Maglicco, 2016; Singapore Ministry of Education, 2012). Underpinning the teaching of mathematics in Singapore is the gradual transition from concrete, to pictorial and finally to abstract application of concepts, and one representation regularly utilised within the pictorial stage is the bar model (Maglicco, 2016; Singapore Ministry of Education, 2012).

2.2.9b Theoretical Framework of the Bar Model

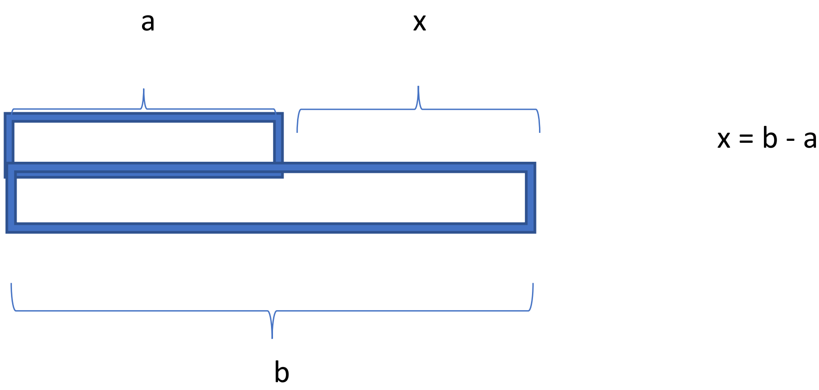
The bar model, or 'heuristic involving diagram or model drawing' (Ng & Lee, 2009, p.284), as a tool for solving both arithmetic and algebraic word problems, is based on the theoretical framework of the processing model for solving arithmetic word problems (discussed earlier) (Kintsch & Greeno, 1985) and was officially introduced into the mathematics curriculum by the Singapore Ministry of Education in 1983 (Ciobanu, 2015; Ng & Lee, 2009). The aim of the bar model is to provide a consistent representational basis for the creation of a visual-schematic diagram that emphasises the relationships within the word problem, in order to deepen pupils' understanding (Maglicco, 2016). Nevertheless, as with any strategy or representation, caution should be applied to ensure that students understand the 'boundaries' of the strategy and 'learn to critique its appropriateness in a given situation' in order to avoid 'overgeneralisation of techniques that do not apply' (Shaughnessy, 1985, p. 408).

Three main types of bar model are commonly used to solve mathematical word problems (Ciobanu, 2015; Mei & Li, 2014) (see Figure 10), on which variation within each of these types, can be used to solve most mathematical word problems.

- Part-whole model



- Comparison model



- Multiplication-division model

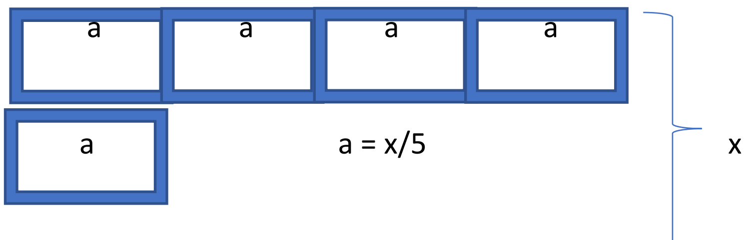


Figure 10: Three common types of bar model application

The approach is based upon the concrete-pictorial-abstract approach to problem solving, which underpins the mathematical teaching sequence within the Singapore mathematics curriculum (Singapore Ministry of Education, 2012). This entails students constructing a schematic representation of the problem (the pictorial stage), which models the mathematical quantities of both the known and unknown values and the relationships between them (Maglicco, 2016). The concrete-pictorial-abstract approach aligns with Bruner's (1966) stages of representation: enactive-iconic-symbolic (discussed earlier). The rationale for this being that if children are provided with the means to visualise (through an external schematic representation) a word problem, the underlying structure of the problem would become more transparent (Ng & Lee, 2009, p. 284).

Application of the bar model approach relies on two key functions to be carried out by the student:

1. Categorisation of the word problem, by type;
2. Drawing the schematic diagram to accurately represent the mathematical relationships depicted in the problem;

(Maglicco, 2016).

Building on from the framework adopted by Kintsch and Greeno (1985) discussed earlier, which focuses on the interaction between comprehension and word problem-solving ability, Mahoney (2012) proposed a theoretical framework which is operationalised through the bar model (illustrated in figure 11) and is based upon Mayer's (1989) two-phase model of problem solving.

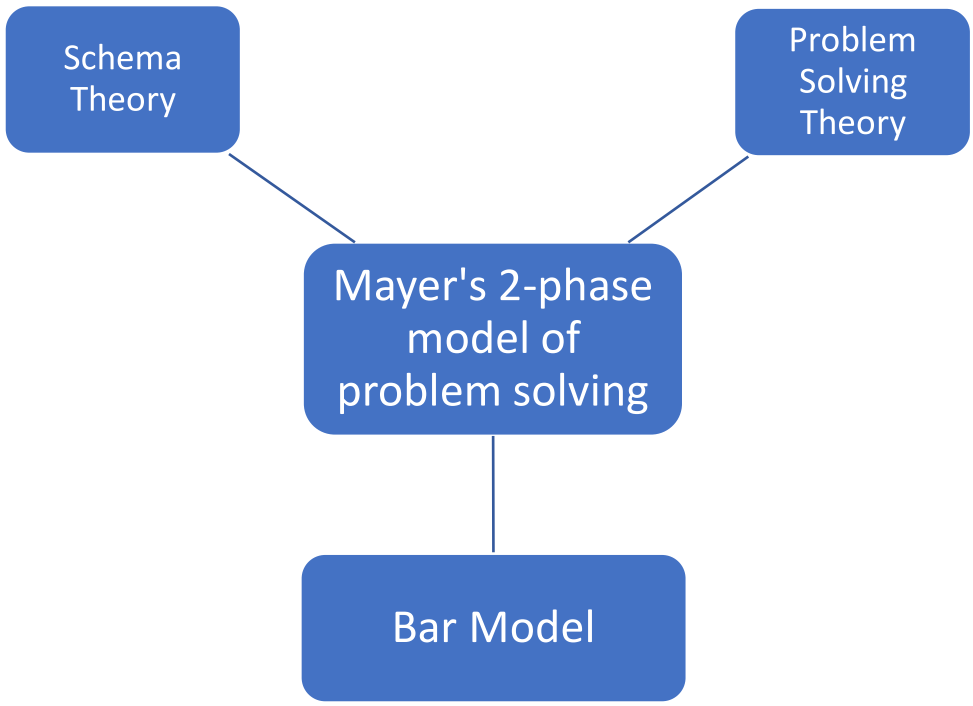


Figure 11: Mahoney's theoretical framework underpinning the bar model approach, based on Mayer's 2-phase model of problem solving (Maglicco, 2016; Mahoney, 2012; Mayer, 1989)

According to Morin et al. (2017), this approach combines both schematic-based instruction (SBI) and cognitive strategy instruction (CSI). SBI is based on schema theory (Maglicco, 2016; Mayer, 1989), where there is a need for students to conceptualise the underlying problem schema. CSI involves building awareness of task demand and direct instruction of problem solving strategies, which it is suggested may address any underlying cognitive and metacognitive deficits (Morin et al., 2017, p. 93).

SBI provides a framework for supporting students through the problem-solving process, as follows:

Identify the underlying structures → select and complete the related schematic diagram → write a number sentence facilitating the solution → carry out the computation → review the solution for reasonableness (Peltier et al., 2019, p. 2*).*

SBI connects the two stages of Mayer's (1989) problem solving process and is based on supporting pupils to draw upon their existing schemas to categorise unfamiliar word problems. As this pedagogical approach relies upon pupils drawing on existing schemas to create a schematic diagram, which emphasises the underlying structure of the word problem, the consistency and fundamental simplicity of the bar model foundational structures may be key to its success. Findings from four systematic reviews and meta-analyses, focusing on the effectiveness of SBI, suggest that such approach can enhance the performance on mathematical problem solving for students both with, and without, disabilities (Cook et al., 2019; Peltier et al., 2018; Peltier & Vannest, 2017; Peltier & Vannest, 2018; Powell, 2011).

Studies have shown the success of the bar model as a tool for supporting individuals with learning difficulties (Maglicco, 2016) and may be a direct consequence of the reduced demands of cognition and working memory (Spooner et al., 2017) required due to the consistent bar representation, thus supporting the rationale behind the current study. Furthermore, many previous studies highlighting the link between SBI and problem solving ability for autistic pupils, have primarily focused on procedural instruction, rather than supporting pupils to develop a conceptual understanding of the underlying structure of word problems (Spooner et al., 2017). Consequently, when it comes to the generalisation of skills required to solve unfamiliar, or novel, word problems, the conceptual understanding of the underlying structures may be lacking. Whilst the bar model may be seen to support problem solving abilities in many learners, it is also worth noting that, in order for this to be successful, other skills, such as conceptual understanding, well-organised knowledge base of number facts and multiplicative reasoning are also required in order to find the correct solution (Ng & Lee, 2009).

In a small, single case design, focusing on advanced ability, 4th Grade (Year 5) students in Nevada, Maglicco's (2016) study found that following eight teaching sessions, delivering instruction and application of the bar model approach, students' scores on problem solving tasks increased from 20% to 92% on multiple comparison problems and 0% to 88% on fraction problems. Their findings suggest that the bar model approach may increase students' performance in problem solving in the areas of multiplicative comparison and fraction word problems. However, despite few studies having explored the bar model approach in isolation, these findings provide an encouraging data set supporting the use of the bar model approach as a tool for increasing students' problem-solving performance.

Much debate exists over whether the bar model serves as a conceptual, or procedural, support for students to solve mathematical word problems (Morin et al., 2017). With the bar model approach, for example, a relational, or conceptual, understanding of the 'part-part-whole' concept is required if students are to be able to correctly select and utilise the appropriate diagrammatic representation of a word problem within a new context, along with instrumental, or procedural, understanding to construct the model.

Despite the claimed successful outcomes of using the bar model to support those pupils with mathematical difficulties in solving word problems (Maglicco, 2016; Morin et al., 2017), it is argued that 'most problem solvers have more difficulty in constructing a useful problem representation than in performing the computations necessary to solve the problem' (Hegarty & Mayer, 1995, p. 19). Thus, as argued by Morin et al. (2017), the use of, and focus on, SBI and CSI may be paramount, if the issues highlighted by Hegarty et al. (1995), are to be overcome. It is worth noting however, that the bar model is one of many approaches to visual representations within mathematics and it is hoped that this approach is not delivered at the expense of current and alternative visual tools to support mathematical understanding.

By considering the theoretical underpinnings of the bar model, principally the influence of problem-solving theory (Polya, 1945), we can begin to establish how the bar model, as a schematic diagram to support mathematical word problem solving, may be an effective approach for supporting the development of the following cognitive strategies:

Paraphrasing (understanding the question)
Visualising (constructing the bar model)
Hypothesising (manipulating the bar model)
Checking (relating the solution back to the question)
(Morin et al., 2017)

Morin et al.'s (2017) study highlights the importance of the theoretical underpinnings of the steps involved in the application of the bar model in terms of the cognitive processes required. This provides a significant platform on which to begin to understand how specific cognitive strengths and difficulties amongst pupils may begin to interact with the application of the bar model as a problem-solving tool.

This section has considered the theoretical underpinnings of the bar model, as a specific type of visual representation. The skills required in mathematical problem solving have been considered against the theoretical foundations of the bar model. The following section provides a summary of the main findings from section 2 of the literature review, before moving on to the final section, which completes the development of the conceptual framework underpinning the current study.

2.2.10 Summary

This section began by attempting to define and understand the concept of word problems as an aspect of mathematical problem solving, along with considering the typical structure of many word problems. Through exploring some key models of word problem solving (Bruner, 1966; Fülöp, 2019; Gningue et al., 2014; Kilpatrick, 1985; Lester & Cai, 2016; Mayer, 1985; Mayer, 1989), it is evident that the skills involved in the process of solving mathematical word problems are multifaceted and may be influenced by a variety of factors. Two key theories – schema theory (Maglicco, 2016; Mayer, 1989) and cognitive load theory (Sweller, 2011; Sweller et al., 2019), appear to play a significant role within this mathematical domain. Through relating some of the skills involved in the process of solving mathematical word problems, Skemp's (1978) theory of relational and instrumental understanding has been discussed, with reference to the pedagogical implications of teaching, and pupils' learning, of mathematical word problems.

Word problems as a unique genre have been explored, along with the implicit rules generated through didactical contracts (Greer, 1997), highlighting the potential influence of the classroom culture on mathematical word problem solving. The ability to distinguish between the reality of events in the real-world and those required for success in the classroom context have been explored and explained through the use of different modes of knowledge (Barmby et al., 2013; Cooper et al., 2018; Deliyianni et al., 2009; Gravemeijer, 2020; Greer, 1997; Hegarty & Kozhevnikov, 1999; Kalyuga, 2007, 2013; Ng & Lee, 2009; Rau, 2017; Van den Heuvel-Panhuizen, 2020; von Glaserfeld, 1987).

The complexity of solving 'real-life' mathematical word problems is beginning to emerge. The potential influence on pupils' word problem-solving ability, of factors such as reading comprehension, executive function skills and an understanding of the nature of reality within the classroom context, coupled with specific pedagogical approaches employed within the classroom, have been explored (Bae et al., 2015; Björn et al., 2016; Boonen et al., 2013; Cooper & Harries, 2002; Greer, 1997; Jones et al., 2009; Kilpatrick, 1985; Morin et al., 2017; Oswald et al., 2016; Özsoy, 2015; Roelofs et al., 2015; Wei et al., 2015; Wen, 2018; Whitby & Mancil, 2009).

Whilst the knowledge that word problem-solving difficulties have multidimensional foundations, the specific effect of different combinations of these factors, coupled with specific deficits commonly found within the autistic population is identified a key gap in current understanding. Moreover, a lack of understanding of the influence of any specific pedagogical tool or approach within these combinations of factors, provides the grounds for the rationale behind the current study.

It is proposed that the use of visual or schematic representations, which are central to the bar model, can assist with the development of new schemas and comprehension of word problems (Kintsch & Greeno, 1985; Maglicco, 2016). In order to overcome the difficulties in forming coherent, relational representations (discussed in WCC above), and the enhancement of memory performance, it is suggested that a clear, organisational framework (of which the bar model may be one example within mathematics) may support this development (Desaunay et al., 2019).

Through drawing upon on the literature reviewed, it is possible to begin to understand the complexity of mathematical word problem-solving, along with some of the potential factors, which may impact upon success in this domain of mathematics for autistic individuals (figure 12) (Bae, 2013; Boonen et al., 2013; Chiang & Lin, 2007; Cooper et al., 2018; Kalyuga, 2007; Keen et al., 2015; Mayes & Calhoun, 2006; Powell, 2011; Rau, 2017; Whitby & Mancil, 2009). The following diagram provides the next step of the development of the conceptual framework for the current study, in order to consider the potentially complex interaction of key conditions within the mathematical word problem-solving process.

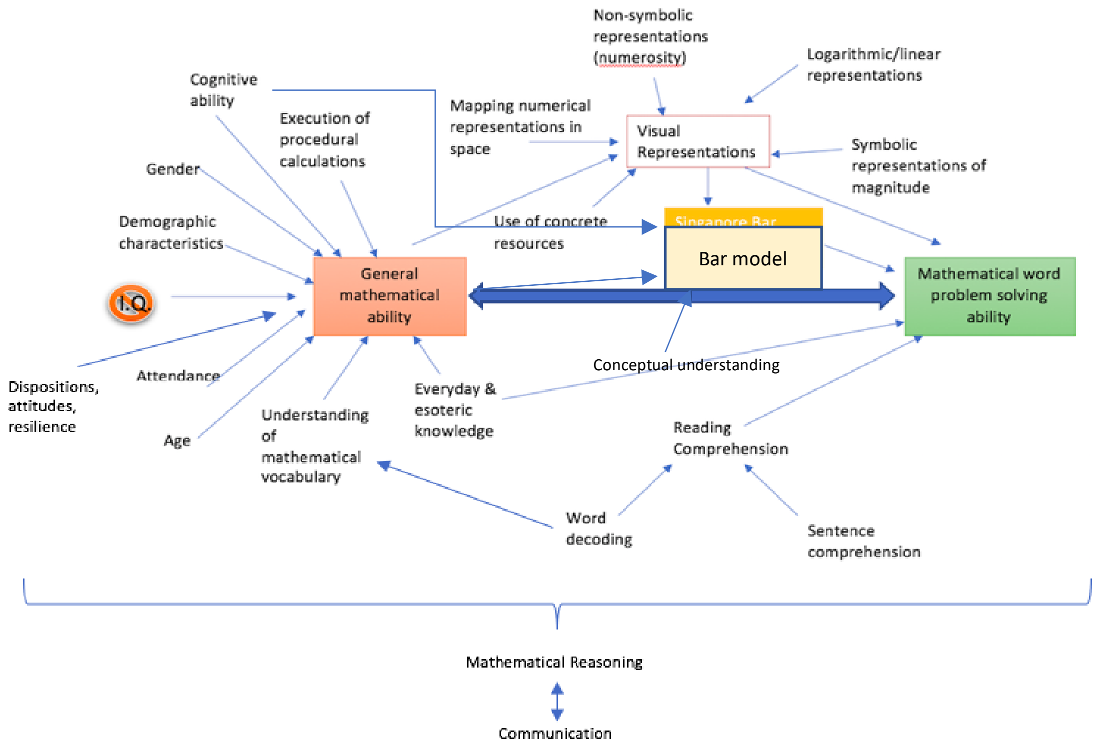


Figure 12: The complexities of mathematical word-problem solving (Thompson, 2019, p. 217)[6]

[6] Although not all these aspects included within this model are discussed in detail, their inclusion adds to the understanding of, and demonstrates the wider complexities involved within mathematical problem solving.

The final section of the literature review now moves on to bring together the findings from the whole chapter. Through doing so, the conceptual framework underpinning the current study is fully developed, to formulate a thorough understanding of the potential conditions associated with mathematical problem solving amongst the autistic population.

2.3 Autism, mathematical problem solving and the bar model: development of a conceptual framework

This final section of the literature review draws together some of the links between the aspects of autism, mathematical problem solving and the bar model, as a specific type of visual representation, to understand some of the mechanisms underpinning these interactions. The findings are used to construct the final conceptual framework underpinning the current study. Through considering the literature discussed, alongside the theoretical underpinnings and application of the bar model, some of the potential underlying mechanisms at play within this complex interaction of attributes and processes are explored. A worked example is used to exemplify the potential interaction of autistic traits with the application of the bar model as a tool for solving a mathematical word problem.

When considering the multifaceted approach to solving mathematical word problems, coupled with the cognitive theories of autism and the phases of applying the bar model approach, we can begin to understand how the heterogeneity of autism, and its associated cognitive deficits, may impact upon mathematical problem-solving ability (see Fig. 13). In the development of the bar model, which represents a macrostructure of the problem, it is suggested that three distinct phases are required:

Phase 1: Understanding the problem – identifying known and unknown quantities through text comprehension;
Phase 2: Structural phase – drawing the model through understanding of the part-whole relationships;
Phase 3: Symbolic phase – translating the model into a mathematical sentence and revision of the model
(Ciobanu, 2015)

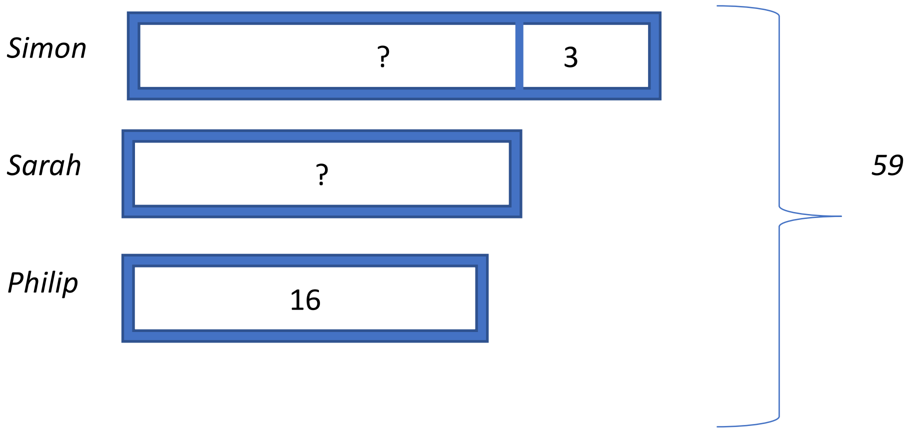
Although similar, in some respects, to the two key functions required by students in applying the bar model, as suggested by Maglicco (2016), Ciobanu's (2015) phases do not explicitly draw attention to the need to categorise the word problem based on students' existing schemas. Nevertheless, it could be argued that this function is embedded within Ciobanu's (2015) first phase – 'understanding the problem'. However, unlike Maglicco's (2016) functions, Ciobanu (2015) considers the requirement to translate the bar model structure into a mathematical meaning, through revision of the model, which is clearly required for students to ultimately work towards the final solution of the word problem.

By considering the specific tasks involved in mathematical word problem-solving (Morin et al., 2017), along with the cognitive and social theories used to understand and explain the perceived behaviours and varied cognitive profiles of autistic pupils, we can begin to develop an understanding of the potential interactions between the two. Furthermore, by considering Ciobanu's (2015) suggested phases of application of the bar model to a worked example, based on the comparison model, (below), we can consider how the bar model, as a schematic representation to support mathematical word problem-solving, may also interact with these mechanisms.

Worked example:

Simon is 3 years older than Sarah. Philip is 18 years old. Two years ago, the sum of their ages was 59. How old is Sarah now?

Model based on two years ago:



? + ? = 59 – 3 – 16 = 40

Therefore, Sarah was 20, Simon was 23 (3 years older than Sarah) and Philip was 16.

Consequently, their ages now (two years later):

Simon 25

Sarah 22

Philip 18

<u>*Solution to the problem: Sarah is now 22 years old*</u>

i. Decoding linguistic information

The verbal and nonverbal implications of ToM deficit within some autistic pupils, may inevitably impact upon the decoding of linguistic information, and understanding of textual cues, presented within the text of word-problem (Berenguer et al., 2017). The literal interpretation of language, often associated with ToM deficits, along with the understanding and interpretation of subjects' actions within the text, may present an immediate barrier to the comprehension of the problem (Kimhi, 2014; Mazza, Mariano, Peretti, Masedu, Pino & Valenti, 2017). Moreover, the need to interpret the 'chunks' of information relating to individual ages within this problem, may indeed be supported through WCC. However, when attempting to consider the global perspective of the problem, (the relationships between the characters' ages and the fact that the sum of 59 relates to two years ago), difficulties in WCC may become problematic at this stage.

By considering autistic traits through the cluster deficit model (Siegel, 2009), it is clear that individuals within the verbal/non-verbal communication clusters, may indeed demonstrate difficulties in processing the sheer volume of linguistic information presented within this question. Furthermore, limitations to working memory, which plays a vital role in mathematical problem solving (Berenguer et al., 2017; Desaunay et al., 2019; Gioia et al., 2000; Goldstein & Naglieri, 2014; Ullman & Pullman, 2015; Utami & Warniasih, 2019; Wen, 2018), may also impact upon the ability to process and comprehend the amount of textual information.

In terms of application of the bar model, Ciobanu's (2015) initial phase requires the understanding of the problem, specifically the relationships and quantities of the known and unknown variables within the problem. Therefore, processing the textual information to understand the relationships between the ages of Simon, Sarah and Philip requires significant cognitive demands. To add to this demand, is the global knowledge that the total of 59 relates to two years ago. Consequently, any difficulties or deficits within this area may have significant implications on the ability to successfully utilise the bar model as a problem-solving tool.

ii. Accurate extraction of the calculation

As central coherence impacts upon the ability to process global and local information (Booth & Happé, 2018; Booth & Happé, 2010; Happé & Frith, 2006; MacLeavy, 2019), WCC may interact with the ability to accurately interpret and extract the required calculations for the problem solution. Any influence of WCC could in fact both support and hinder this stage of the problem-solving process. WCC tends to enable individuals to demonstrate superior abilities at processing local information (Booth & Happé, 2018; Booth & Happé, 2010; Happé & Frith, 2006; MacLeavy, 2019), which may indeed provide, or even prove, beneficial in terms of identifying the detailed cues within the text to allow for the calculation to be extracted (individual ages and relationship between Simon and Sarah). On the other hand, as WCC tends to give rise to difficulties in processing global information, the lack of the overall global context of the word problem, may also hinder this stage, particularly when considering the information relating to the sum of their ages was based on two years ago. Here, Philip's age may be particularly problematic, as his age of 18 relates to now, yet the sum of 59 relates to two years ago.

iii. Retrieval of mathematical facts

Linked closely to both long-term and working memory, the retrieval of mathematical facts may be influenced by WCC or deficits within the executive functions (Berenguer et al., 2017; Gioia et al., 2000; Goldstein & Naglieri, 2014; Ullman & Pullman, 2015). Again, the need to process information at the global level of the problem, along with the demands on working memory and the requirement of organised search for fact retrieval may present difficulties at this stage of the problem-solving process.

Once again, using the word problem example given above, the demands on working memory can be seen to be substantial. Consequently, a student demonstrating difficulties within the EF skills may, with such a question, present difficulties with organising and processing this volume of information. However, in a synthesis of 65 studies, on pupils demonstrating mathematical difficulties, the effects of cognition and EF (specifically working memory) yielded mixed results (Powell et al., 2019), so consequently should be treated with caution and considered carefully according the data within the current study. Coupled with the requirement to draw on existing knowledge schemas in order to retrieve the required facts for this problem solution (Maglicco, 2016; Marshall, 1995; Mayer, 1985; Peled & Wittrock, 1990; Schoenfeld & Herrman, 1982), may place significant cognitive load on top of the load already experienced through attempting to organise and process such a large volume of information.

iv. Application of mathematical concepts

Once more, the demands on working memory required to apply mathematical concepts, may present a barrier to those individuals with difficulties in the EFs. The ability to plan and direct one's cognitive strengths to the application of the mathematical concepts within the word problem above may become impaired through EF deficits (Goldstein & Naglieri, 2014; Lecce et al., 2019; Polya, 1945; Wen, 2018; Ziermans et al., 2017). Similar to the discussion above, an already significant cognitive load is required for the previous stages, thus the ability to add to this and direct cognitive strengths to the application of the mathematical concepts may indeed be further hindered through difficulties with EFs.

v. Creating Representations

Any impaired ability to comprehend and represent the global information within the word problem, may impact upon the ability to create an accurate representation of the mathematical information extracted from the problem. Significant cognitive demands are made here, requiring decisions to be made as to the most appropriate representation, the relationships between the known and unknown quantities of the characters' ages within the problem, and the process of constructing the representation. Difficulties within EFs are of clear significance within these tasks, in terms of organising the information and planning the steps required to construct the model.

Additionally, those individuals with deficits in the play/exploration cluster (Siegel, 2009), may have difficulties drawing on under-developed schemas for the choice of visual representation, along with restricted learning through the exploration of the visual representation to take place. Here, the application of the bar model, as a consistent representation (Maglicco, 2016), may reduce the cognitive load on the working memory demands of the individual, through enabling the processing of information to be distributed between verbal and visual channels of processing – particularly in terms of representing the relationships between the known and unknown ages of the characters within the problem (Cooper et al., 2018). Furthermore, due to its consistent basic structure, the bar model may support the development of schema associated with the creation of the visual representation.

vi. Identification and execution of procedural operations

Again, impairments with the EF skills may impact upon this final stage of the problem-solving process. The limitations of the potentially already cognitively overloaded working memory, coupled with the planning and decision-making processes required to execute the procedural operations, may present difficulties within this stage. The need to consider the global context of the problem (that the sum of 59 was relevant two years ago), difficulties associated with WCC may impact upon this final solution phase.

In terms of deficits within the social cluster (Siegel, 2009), the tendency to apply instrumental and incidental learning to this stage, may hinder the finding of a context specific solution to the problem.

The final phase of application of the bar model, the structural-symbolic phase, again may reduce the demands on working memory through enabling information to be processed through both verbal and visual channels (Cooper et al., 2018). In addition, the consistent underlying structure to the bar model, may support the process of manipulating the representation to support the solution finding stage of the problem-solving process (Maglicco, 2016).

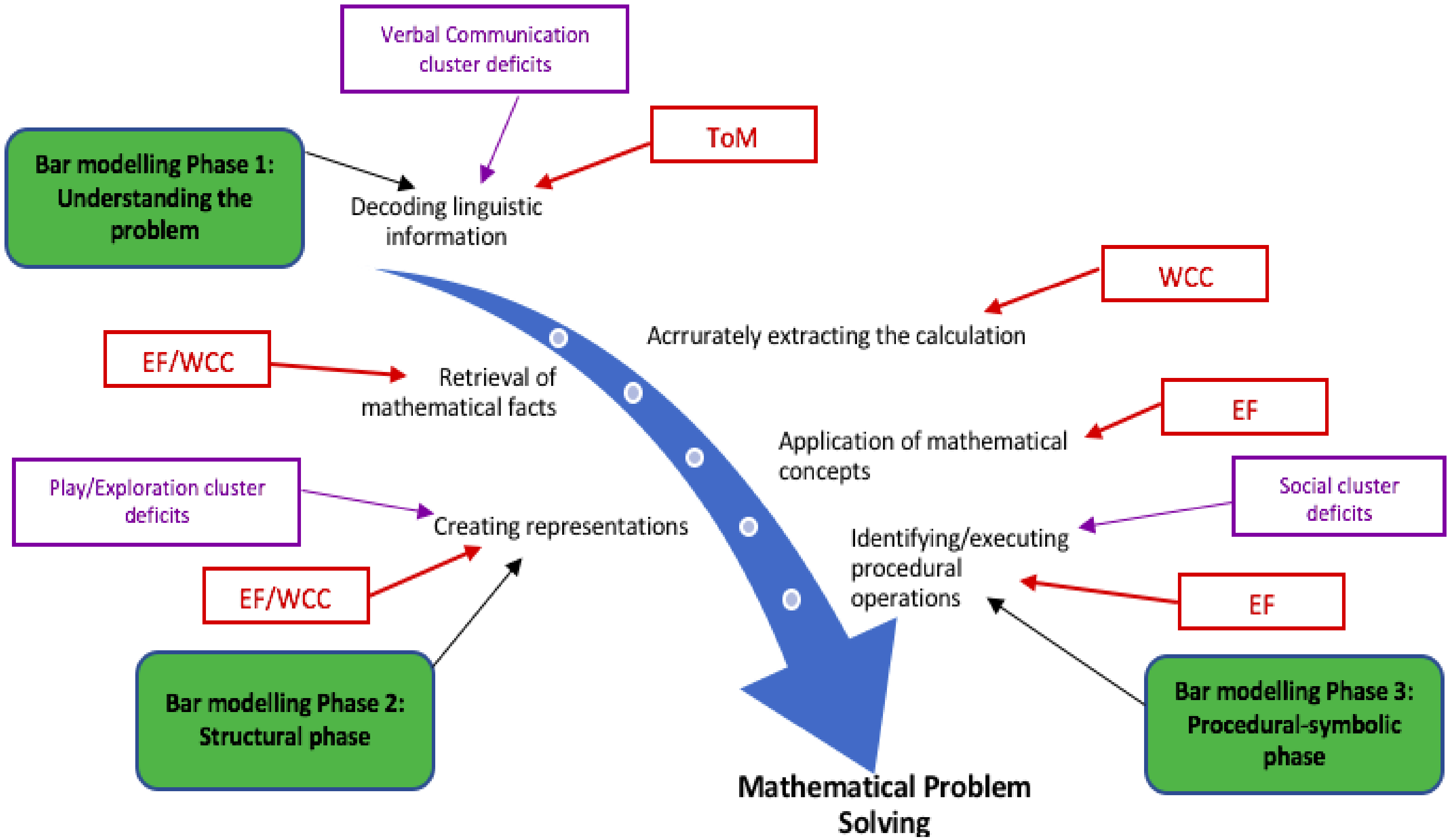


Figure 13: The multifaceted tasks involved in mathematical problem solving (based on (Morin et al., 2017, p. 92) and potential associated implications of cognitive theories and cluster deficits (Siegel, 2009) underpinning autism, and how Ciobanu's (2015) phases of bar modelling can be applied (Thompson, 2019)

Although many of the difficulties discussed in the worked example above may also be apparent in many NT children attempting to solve such a problem, the specific associations with ToM, WCC and EFs, along with the cluster deficits (Siegel, 2009) discussed, may exacerbate these difficulties for autistic pupils.

In consideration of Figure 13., and the associated difficulties faced by those individuals on the autistic spectrum, Morin et al. (2017), suggest that any difficulties in the areas of reading (decoding or comprehension), cognitive processes and mathematical cognition, may all hinder the word problem-solving process. However, in contrast, Ngeno et al. (2019), suggest that an individual's ability to read mathematical texts is not indicative of successful mathematical word problem solving.

When considered against the processing model for solving arithmetic word problems (Kintsch & Greeno, 1985), WCC theory could be significant. Whilst Kintsch and Greeno (1985) suggest that in order to solve a word problem, pupils must first abstract key information from the text into 'chunks', potentially being supported through the attention to local detail through WCC, the step of model drawing and representation, of which the bar model is one approach to this, requires the information to be contextualised within the macrostructure of the word problem, which may be affected by difficulties in global processing for those individuals with WCC (Kintsch & Greeno, 1985, pp. 290–291).

Furthermore, it is suggested that those students who have weaknesses in general reading or mathematical skills, can benefit significantly from the bar model approach, as a consequence of following a step-by-step approach within the process (Hoven & Garelick, 2007). Nevertheless, it must also be considered that any underlying reading difficulties, may potentially hinder the pupils' understanding of the construction of the bar model. However, it could be argued that the bar model approach is only one approach to solving word problems through a step-by-step process, thus this does not necessarily suggest that it is the specific bar model approach, which supports these individuals, but rather the sequential mechanism underlying this process.

At the core of mathematical problem solving, both within the autistic and NT population, seems to be the potential influence of a variety of skills and abilities:

- Arithmetic skill and computational ability;
- Inhibition of alternative behaviours;
- Non-verbal problem-solving skills;
- Reading fluency and comprehension;
- Concept formation;
- Cognitive load required within each set schema;
- Working memory capacity;
- EF skills;
- Verbal ability and language;
- Spatial ability;
- Metacognitive factors, such as willingness, perseverance and self-perception;
- Prior mathematical knowledge;
- Ability to construct visual representations;

(Boonen et al., 2013; Deliyianni et al., 2009; Fuchs et al., 2006; McLeod, 1985; Ngeno et al., 2019; Peltier & Vannest, 2018; Rosli et al., 2013; Schoenfeld, 1985; Silver, 1985; Siregar et al., 2019).

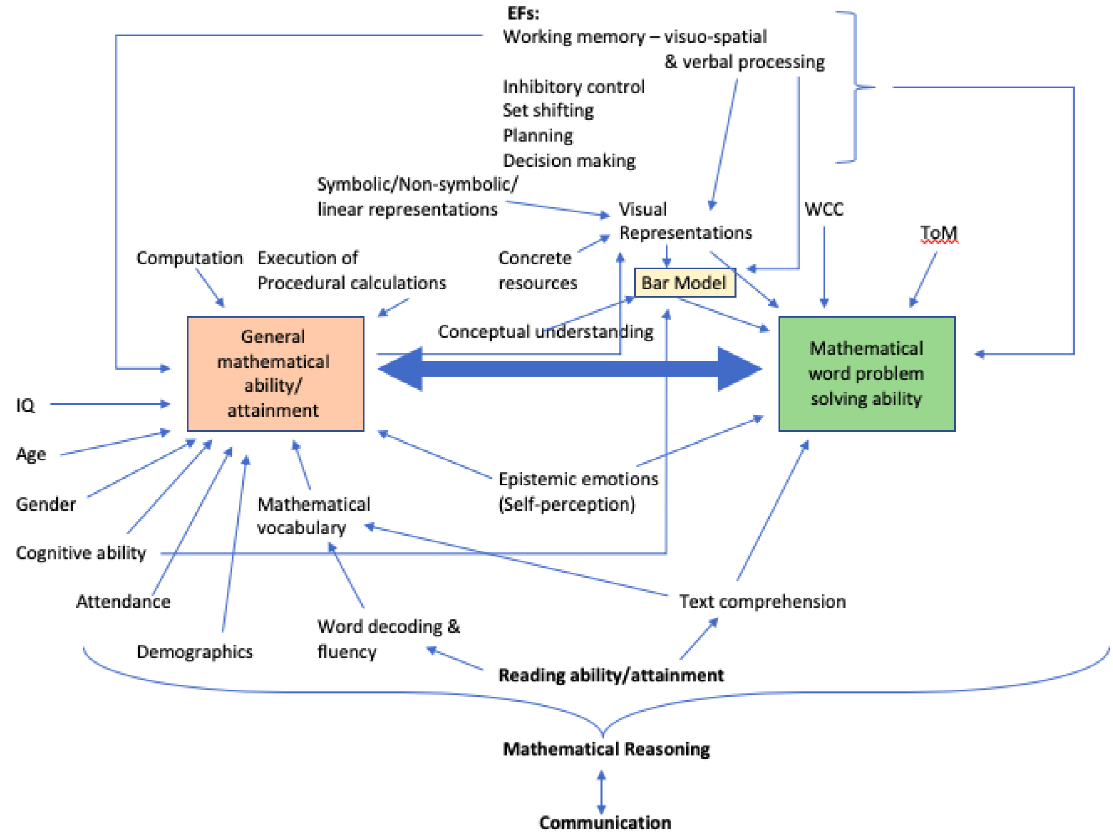


Figure 14: The conceptual framework underpinning the current study

Figure 14 above provides the conceptual framework underpinning the current study to consider the potentially complex interaction of key conditions within the mathematical word problem-solving process. Through considering these conditions, along with cognitive factors associated with autism, specific combinations of conditions can be explored, to identify pathways containing specific combinations of conditions which yield successful problem solving amongst the autistic population. The conceptual framework is used to guide the selection of conditions for analysis within the current study (see chapter 3.4.2).

Thus, in summary, the review of literature within the themes of autism, mathematical problem solving, and visual representations can be summarised as follows:

- Mathematical problem solving is a multifaceted process and is potentially influenced by a number of conditions (factors);
- The conditions may interact in different combinations to support mathematical problem solving;
- The varied cognitive profiles of autistic individuals, and the associated behaviours and abilities of these individuals, are mediated through a complex set of underlying mechanisms;
- The use of visual representations as a tool for supporting mathematical problem-solving show varied results according to the type of representation used, and the conditions associated with different sub-groups of individuals;
- The bar model, as a schematic representation, provides a consistent foundational model, in which the relationships between known and unknown variables within a mathematical problem can be represented and then manipulated;
- When considering the conditions potentially impacting upon mathematical problem-solving performance, problem-solving ability within autistic individuals, and application of the bar model as a schematic representation, the relationships between all these factors may interact to produce differing outcomes of success.

Despite the literature providing a comprehensive insight into the range of factors associated with successful problem solving in mathematics, a significant gap in research is evident as to the interactions, and combinations, of these factors, when coupled with the behaviours and difficulties commonly associated with autism, and the overall impact upon problem solving ability. Furthermore, more research, carried out within educational settings, is needed into the specific factors associated with problem solving processes amongst the autistic population in order to support the development of effective classroom practices (Bae, 2013; Chiang & Lin, 2007; Wallace et al., 2019). Specifically, in addition to the requirement of further research into the combinations and interactions of these factors, is the gap in research as to the effectiveness of the bar model, as a visual representation, to support mathematical problem solving. Finally, when it comes to research involving autistic individuals, there is a lack of empirical research within the context of the classroom, to identify successful strategies and approaches to support autistic individuals with mathematical problem solving (Doobay et al., 2014; Wallace et al., 2019). Consequently, the current study seeks to address these gaps in the research through exploring the conditions, and combinations of conditions, underpinning successful problem solving within the mathematics classroom. The current study focuses on the use of the bar model, as a specific type of visual representation in mathematical problem solving, to ascertain its likely effectiveness and role within the execution of mathematical problem solving.

Chapter 3: Methodology

This chapter begins by positioning the current study within the philosophical framework of critical realism. The concept of causal mechanisms, operating within the domains of the real and the actual, which give rise to events observed within the domain of the empirical are discussed. Maintaining the philosophical underpinnings of critical realism, the methodology of qualitative comparative analysis (QCA) is then introduced to the reader. QCA is discussed both as a research design and a research technique within the current study.

The data collection techniques used within the current study are discussed before presenting the details of the preliminary study. The findings from the preliminary study are used to drive the refinement of the research design and, more importantly, the calibration of the condition measures used within QCA for the final analysis.

Finally, the data analysis techniques, used within the current study, are discussed in detail. A comprehensive account of the use of data analysis within QCA is presented to provide clarity and robustness to the study.

3.1 Critical realism as a philosophical framework

The research questions and research design within the present study are addressed through a critical realist (CR) lens, which has become a more popular philosophical framework over recent years within the social sciences. Positioning itself between the epistemological poles of positivism and relativism (ontological realism and epistemological relativism) (Groff, 2004), the foundations of CR pave the way towards a mechanistic explanation of causality (where causal generative mechanisms are responsible for producing a given outcome). Such meta-theory is important for use in research, as data collection and analysis is always underpinned by ontological and epistemological assumptions and can therefore not be a-epistemic (Scott, 2014). As such, CR provides an ontological frame of reference, in which reality is considered as structured into three levels to provide ontological depth: the domains of the empirical, the actual and the real. Within the domain of the empirical, events are experienced by ourselves and perceived through human experience (for example, I saw [or heard] a tree fall down). In the domain of the actual, events occur whether we experience or interpret them ourselves, or not (the tree made a sound, even though there was nobody there to hear it). Finally, the domain of the real, where causal mechanisms exist (root rot, age or high winds caused the tree to fall down) (Anderson, 2019; Archer et al., 2007; Fletcher, 2017; Pawson & Tilley, 1997; Scott, 2014).

Based on this structuring of reality, it is considered that the generative mechanisms and structures – the focus of CR (MacLeavy, 2019), which give rise to observable phenomena, operate at a level independently of the domain of the empirical. Consequently, reality is considered as being structured to provide ontological depth:

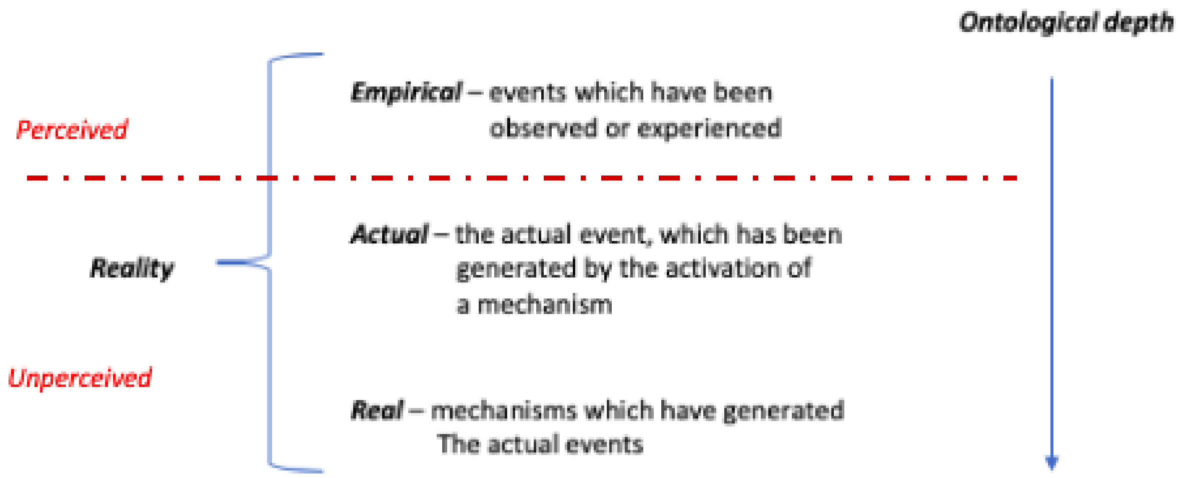


Figure 15: The structured levels of reality from the perspective of ontological realism, identifying those domains, which are perceived, and those which are unperceived (developed from Anderson (2019))

Therefore, from a CR perspective, it is suggested that causal mechanisms, which operate within the domain of the real, may be studied and understood through phenomena observed at the empirical level.

It should be noted here that mechanisms are not simply regularities, but are ‘potentially causal generative processes’, which can operate in specific contexts and conditions, as discussed above (Jones, 2010, p. 203). Based on this notion, the current study seeks to uncover and explore the underlying mechanisms (associated with mathematical problem solving, cognition and autism) operating within the domain of the real, through observing mathematical problem solving amongst autistic pupils within the domain of the empirical. Furthermore, the study explores the conditions, and combinations of conditions, under which these mechanisms are actualised and give rise to the empirical observation of successful mathematical problem solving for this group of pupils. From this perspective, causality is ‘understood as a tendency of objects, which may or may not be actualised’ (Scott, 2014, p. 34). Based on this understanding, due to ontological depth, objects within the systems have powers. As such, these powers may indeed be present, but only actualised under certain [specific] conditions, where the causal influence is actualised within the open system. Hence reference to the term ‘tendencies’, as such powers, if not actualised through the influence of specific conditions (or other powers), may be present, however dormant (and therefore not experienced).

When considering the structure of reality, it is useful to consider Bhaskar’s (2007) representation of where mechanisms, events, and experiences within the social world, align with the layers of reality (table 3).

	Domain of Real	Domain of Actual	Domain of Empirical
Mechanisms	X		
Events	X	X	
Experiences	X	X	X

Table 3: The intersection of mechanisms, events, and experiences with the structured layers of reality. From (Archer et al., 2007, p. 41)

Consequently, it can be argued from this perspective, that science is seen as a social product. However, the mechanisms underlying the phenomena, which operate within the domain of the real are independent of the discovery and observation of such a phenomenon operating within the domain of the empirical through unobservable powers and mechanisms (Danermark et al., 2002). In turn, such mechanisms may exist and continue to operate, giving rise to events within the domain of the actual, despite the absence of the observation or discovery of the phenomena within the domain of the actual (Archer et al., 2007).

Use of the term 'critical' (or in Bhaskar's (1975) terms 'transcendental'), in part, refers to the simplified empiricist view of ontology, where phenomena are studied at the empirical level only, and the domains of the actual and the real are not considered. Furthermore, it is critical of universal claims of truth, often encountered within positivism. The 'realist' aspect within CR is based on the notion of the existence of a reality, operating independently of our knowledge or observations of it (Danermark et al., 2002). Based on this assumption, and once again providing an argument against positivist perspectives, the inability of any research to control the real world, such as the social structures within the classroom in relation to the current study, must be considered. If research could, in some way claim to control the open systems of the real world, then any results could not be predictive of the complexities in existence within the real world (Anderson, 2019; Fleetwood, 2017). This argument supports the gap in current empirical research (discussed earlier), to identify successful strategies and approaches to support autistic individuals with mathematical problem solving within the context of the classroom – an open system (Doobay et al., 2014; Wallace et al., 2019).

From a critical realist perspective, there is a clear distinction between ontology and epistemology (Archer et al., 2007; Maxwell, 2012), where 'ontology is not reducible to epistemology' (Fletcher, 2017, p. 182). From this perspective, it is argued that reality has an 'objective existence' (Danermark et al., 2002, p. 15) and that our knowledge of such reality is based on 'socially determined conceptual constructions' (ibid. p.17). This highlights the shift that CR makes from that of empirical realism, where the only objects considered to be in existence (or to have existed) are those which have been perceived (Groff, 2004).

Fletcher (2017) explains how the world is ‘theory-laden’ rather than ‘theory-determined’ within a critical realist framework. Hence, in line with the structured view of reality discussed earlier, it is epistemologically plausible for theories to exist, which are yet to be discovered or described. Consequently, the aim of the present study, is to explore and identify the underlying mechanisms, which occur within the domain of the real, to gain knowledge of the phenomena they produce, with respect to mathematical problem solving amongst autistic pupils, which, in turn, can be observed within the domain of the empirical. As proposed by Scott (2014), such research must involve not only the identification of the generative mechanisms, but also the conditions under which the effects of the mechanisms have been observed, which the current study seeks to identify.

From a critical realist perspective therefore, epistemologically, it is believed that no theory is complete or infallible, as it is quite possible for events to exist where no current theoretical knowledge is present (or even possible) (relating to the ‘realism’ aspect of CR). Again, this supports the distinction between epistemology and ontology discussed above (Archer et al., 2007). Thus whilst critical realism underpins this research, it is important to remain open to theory development throughout (and beyond) the current research (Maxwell, 2012), through the utilisation of iterative (back and forth) methods of data collection and analysis (discussed below).

Based on the ontological and epistemological foundations of critical realism, the notion of causation is based upon the interaction and configuration of three constructs: context, mechanism and outcome – often referred to as CMO (context-mechanism-outcome) configurations.

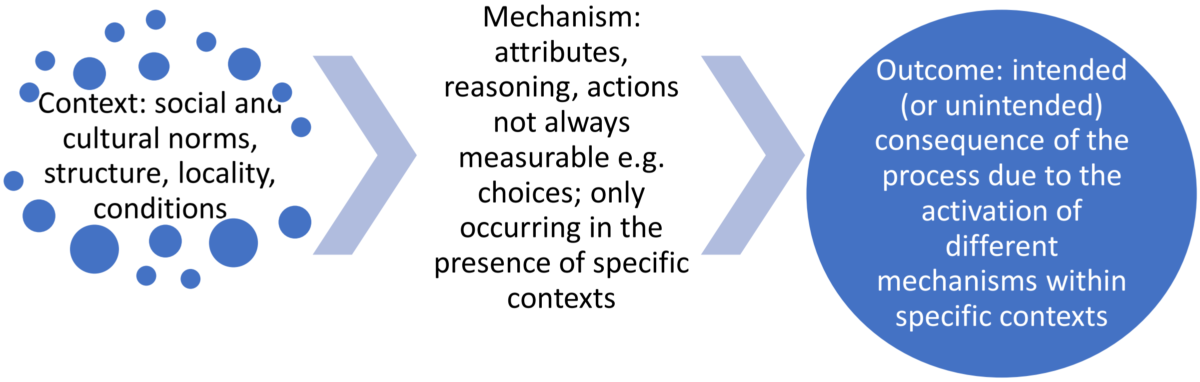


Figure 16: A realist view of causal explanation, based on the configuration of context, mechanism and outcome

Within CR research, considerable emphasis is placed on the context-dependence of causal explanation. The context within a causal process is intrinsically involved in that process, and often cannot be 'controlled for' in a variance-theory sense without misrepresenting the causal mechanism. As a result, the social contexts of the phenomenon studied are [therefore] crucial for understanding the operation of the causal mechanisms (Maxwell, 2012). Hence, the current study draws significantly on the background data for each of the cases (pupils) (discussed in chapter 3.4, below), to provide an in-depth understanding of their context. This is important, as the mechanisms (which are 'real', and which the world consists of), rather than the events (which are 'actual'), operate in conjunction with one another to produce the 'phenomena that constitute the actual […] happenings of the world' (which can be experienced at the empirical level – in the case of the current study - successful mathematical problem solving) (Bhaskar, 2008, p. 47). This is highly significant in the present study, as case- and context-dependent factors (referred to as conditions), such as age, gender, pupils' level of mathematical understanding and socio-economic background, amongst others, may impact upon the observed outcome of successful problem solving in mathematics.

From this critical realist perspective, it is not simply a matter of suggesting causation as a series of observable events, but the contextual factors underlying the causal mechanisms to be intimately involved within the causal processes must be considered. Unlike a variance theory approach to causality, as in randomised controlled trials (RCTs), this view is known as 'process theory' (Mohr, 1982), where emphasis is placed upon the causal processes and consideration is given to how some events influence others (Maxwell, 2012). As such, Maxwell (2012) justifies the ability to make causal claims based on single cases, avoiding the need for control groups and experimental approaches, supporting the justification of a small number of cases (N=9) in the current study. However, caution should be applied here, as it could be argued that the causal processes may be influenced by contextual factors, which may be different amongst different cases, thus naturally preventing such generalisations.

Consequently, it may be more appropriate to consider Bassey's (2001) concept of 'fuzzy generalisation'(p.5). In considering the complexity of the social world and postpositivist epistemological perspectives, which view meaning as contextually-dependent, Bassey (2001) suggests the concept of 'fuzzy generalisations' as a more appropriate way of drawing 'usefulness' from social research findings (Bassey, 2001). Unlike scientific generalisations, which usually suggest that a particular event gives rise to a specific consequence, Bassey's (2001) slight change in linguistic use, proposes that fuzzy generalisations are expressed in the form of 'particular events may lead to a particular consequence' (p.6) – those generalisations made are 'true in most situations, but not necessarily all' (p.9). Aligning with the methodological approach of qualitative comparative analysis (QCA) (discussed below), fuzzy generalisations set out to describe the 'conditions under which a particular phenomenon may, or may not, occur' (Bassey, 2001, p. 9), which is dependent upon the CMO configurations discussed earlier. Consequently, the current study seeks to identify the specific conditions under which successful problem solving may occur within autistic cases. Thus, through the support of in-depth background data (discussed in Chapter 3.4), fuzzy generalisations may be made based on the causal processes inferred from a small number of cases, as with the current study, where N=9 (Bassey, 2001; Maxwell, 2012). Hence, causality 'concerns specific configurations that are temporal and local, which activate certain mechanisms that bring about specific reality' (Gerrits & Verweij, 2013, p. 175). Consequently, the aim of the current research is to seek the best explanation of reality through engagement with the existing theories provided by other researchers (Boonen et al., 2013; Bruner, 2006; Gningue et al., 2014; Skemp, 1978), which may be considered as fallible (Fletcher, 2017, p. 186).

As argued by Danermark et al. (2002), beyond whatever research methods are applied in the search for our knowledge of the truth (or reality), is that of language. However, she also heeds caution within this statement, as the meaning associated with language 'is never definite or fixed' (p.27). Consequently, within the present study, analysis of language used by both the pupils (cases) and their teachers (contributors to building the picture of the case), plays a key role in the search for the underlying mechanisms at play in terms of providing an explanation for the observed phenomena (see chapter 3.6.1). Furthermore, when analysing the data from the current study, in particular the linguistic features used by the pupils and teachers, the concept of 'double hermeneutics' must be borne in mind (Danermark et al., 2002; Isaksen, 2016). Here, the researcher must attempt to interpret the meaning behind the language used, which in turn, is an interpretation itself, as 'our knowledge of reality is filtered through language and concepts that are relative and changeable in space and time' (ibid. p39). Hence, the use of teacher interviews, and the interpretation of such data, to provide background data on the cases (pupils) (discussed in Chapter 3.4) is crucial.

3.2 Qualitative Comparative analysis (QCA) and critical realism

The current research sets out to establish whether the bar model, as a visual representation, supports pupils' ability to solve real-life word problems in mathematics through exploring any sufficient[7] or necessary[8] sets of CMO configurations. Consequently, the approach taken within the present study will be that of QCA, which aims to establish the mechanisms responsible for the phenomena of problem-solving ability and the associated set of specific conditions for autistic pupils.

As an overarching research design, the theoretical foundations of QCA align firmly with the critical realist (CR) framework of causation discussed above (Jopke & Gerrits, 2019). As discussed earlier, from a CR perspective, those observed behaviours and outcomes (the empirical) are determined by underlying mechanisms, which are not observable (and therefore may not be able to be measured) operating within specific contexts. Such CR research seeks to explore these mechanisms and the contexts within which they are activated (CMO configurations), to provide an explanatory view of causation. Aligning with CR, QCA, as a methodological approach, sets out to explore the multiple complex configurations of conditions (which may be contextual or mechanistic) necessary to give rise (or not) to a specific outcome (or action)(in the case of the current study – successful mathematical problem solving) (Anderson, 2019).

3.3 QCA as a methodology

Sociologist Charles Ragin (Ragin, 1987), developed Qualitative Comparative Analysis (QCA) - a 'configurational comparative method of causal inference' (Thiem, 2018, p. 1), as both a research approach and technique (Rihoux, 2013; Schneider & Wagemann, 2010). Although applied to empirical research across a range of disciplines, it is most widely used within

[7] As explained in the introduction, and discussed in more detail later, to support the reader, these terms are simplified here: a 'sufficient' condition 'always occurs when the outcome is present. However, the outcome could also result from other conditions.'
[8] A 'necessary' condition is 'always present when the outcome occurs (i.e. the outcome cannot occur in the absence of the condition)' (Rihoux & Ragin, 2009, p. xix).

political science and comparative sociology, (Berg-Schlosser, 1998; Collier, 1999; Devers, Kelly. et al., 2013; Esping-Andersen, 1990; Harriss-White et al., 2013; Roig-Tierno et al., 2017), where 'entire societies, economies […] or states' (Rihoux & Ragin, 2009, p. 3) have often been the focus of the study. Despite beginning to be used more widely across other disciplines, particularly from the late 2000s onwards (Devers, Kelly. et al., 2013; Rihoux, 2013; Roig-Tierno et al., 2017), including education (Byrne, 2013; Cooper & Glaesser, 2018; Roig-Tierno et al., 2017), QCA is still relatively underused, particularly within educational research. This gradual increase in use within empirical research may be due to its well-suited approach 'for building empirically founded theories that emphasise causal complexity' (Thiem, 2018), seeking to identify causal pathways to an outcome of interest (Hirzalla, 2020; Rihoux & Ragin, 2009; Thiem, 2018).

An increase in the number of published QCA studies continues to appear across disciplines, (Roig-Tierno et al., 2017; Thiem, 2014), including some diversification of topics. Between 1987-2001, the number of studies using QCA was low, due to its novelty. This number has increased rapidly, from 5 in 2001, to 134 in 2014-15 (Roig-Tierno et al., 2017, p. 19). However, many of these still remain limited to the field of political science and public policy analysis – the discipline in which it was primarily designed (Rihoux, 2013; Roig-Tierno et al., 2017). Only recently, has QCA begun be used within the discipline of education and has emerged within standard educational research textbooks (Cooper & Glaesser, 2018; Roig-Tierno et al., 2017), thus being more widely considered as an approach within this field.

In a bibliometric analysis of 469 global articles (accessed in 2016) only 17 were found to be within the field of education, compared to 100 in comparative politics, 85 in business and economy, and 72 in sociology (Roig-Tierno et al., 2017, p. 19). Consequently, a 'high level of standardisation', in terms of the evaluation of robustness has not yet been established (Thomann & Maggetti, 2017, p. 4) and furthermore, 'several authors have recently argued that QCA has, in fact, never been thoroughly benchmarked against common standards of evaluation for methods of causal inference' (Baumgartner & Thiem, 2017, p. 2). Although there are several variants of QCA (crisp-set QCA; fuzzy-set QCA; multi-variate QVA), which are discussed later in the chapter, the current study focuses on fuzzy-set QCA, as a means of quantifying the qualitative conditions under study.

Due to the relative novelty of QCA as a research approach, particularly within the field of educational research (Cooper & Glaesser, 2018; Rihoux & Ragin, 2009; Roig-Tierno et al., 2017), this chapter provides a much more extensive review of the approach for the reader. Furthermore, due to the complexity of QCA in guiding the overall research design of this study, a significant proportion of the current chapter is dedicated specifically to discussing the technical and theoretical underpinnings of the approach (chapter 3.3.2). Thus, QCA is discussed as a central aspect to the research design, providing the reader with a further understanding of both the ontological and epistemological foundations (discussed earlier), as well as the implications of Boolean logic, on which QCA is grounded. Within the current study, consideration of the rationale behind the case selection and the choice of conditions (variables, or factors) to be analysed (see Chapter 3.4.1 and 3.4.2) is of great significance, as the transparency and clarity over such decisions are key to the robustness of the approach (Rihoux & Ragin, 2009; Schneider & Wagemann, 2010; Schneider & Wagemann, 2013).

As a technique, QCA aims to explore set relations of conditions in terms of sufficiency and necessity for a particular outcome (discussed above), and can be either case- (where the unit of analysis considers the case as a whole, e.g. an individual or an organisation) or condition-oriented (where the units of analysis consist of specific phenomena, e.g. self-perception, individuals' socio-economic background, measures of academic attainment for individuals, and so on). Unlike variable-oriented approaches, the use of a case-oriented approach within QCA, placing a key emphasis on complex interactions, enables the study of the 'complex interaction' of conditions (factors) within a relatively small number of cases (Thomas, 2011, p. 512), as in the current study. However, to ensure the robustness of case and condition selection within QCA (discussed in detail in Chapters 3.4.1 and 3.4.2), justification for those conditions (factors) of interest, like the cases themselves, is made and based upon theoretical knowledge (Hirzalla, 2020; Rihoux & Ragin, 2009; Schneider & Wagemann, 2010; Schneider & Wagemann, 2013). As a consequence of this 'holistic approach to causal data analysis', QCA is popular within research based upon theories which consider complex conjunctions of conditions and events (Thiem, 2014, p. 487), as in the present study. Consequently, case-oriented QCA enables the in-depth case knowledge, in conjunction with cross-case inference, to strengthen the internal validity of the research.

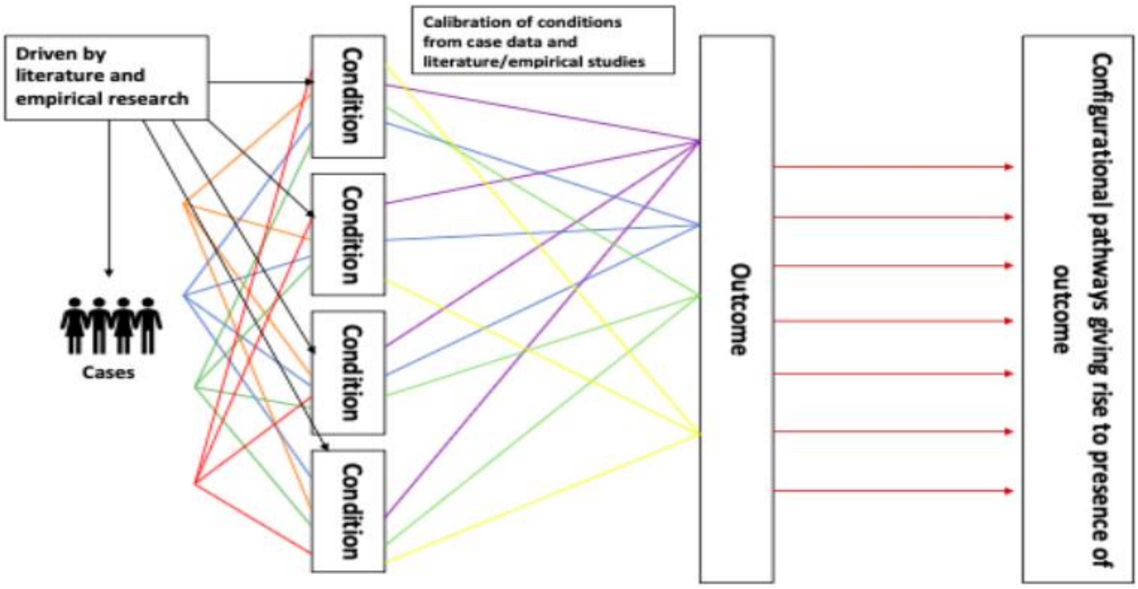


Figure 17: A diagrammatic representation of QCA, as a methodological approach

Figure 17 above, illustrates a simplistic representation of QCA. The degree to which each of the cases meet the conditions (factors) (set membership) is established through in-depth case knowledge. For example, in very simplistic terms, in relation to the current study:

> Condition (factor): using the bar model to solve a mathematical word problem;
> Case 1: Uses the bar model to solve the mathematical word problem;
> Case 2: Does not use the bar model to solve the mathematical word problem;
> Then, case 1 shows full membership (or presence) of this condition (represented by 1) and case 2 shows full non-membership (or absence) of this condition (represented by 0).

Following on from this, all possible combinations (configurations) of conditions, which give rise to the outcome of interest, are considered. Consequently, through analysis of the complex configurations of all conditions (as in the example above), pathways, which give rise to the presence of the outcome of interest, can be established. Through combining this process with the in-depth case-based knowledge of each of the cases, the researcher can understand the complex configurations of conditions (and possible associated contextual, or mechanistic, factors) required, to give rise to the outcome of interest. Therefore, what CR seeks, and what QCA offers, is an understanding of the causes of an outcome (or action) when the conditions are right, and the generative mechanisms underlying the observable outcome (Anderson, 2019; Harré & Madden, 1975; MacLeavy, 2019).

3.3.1 QCA as a research design and approach within the current study

Unlike positivism, where it is assumed that the social world mimics the natural world, and poststructuralism, where the social world is assumed to be 'discursively constructed', CR acknowledges the significance of complex, open systems, along with the way such systems are perceived by humans (Anderson, 2019; Fleetwood, 2017). Both CR and positivism are based on the same assumption that social truths exist independently from one individual's perception, however, whereas positivism considers influences in closed systems, CR considers the social world as an open system (Anderson, 2019). Therefore, the social world (and therefore research carried out within the social world, as in the present study) is an open system – subject to numerous underlying mechanisms, which may or may not be active at any given point. Within such an open system, the social and cultural contexts which give rise to supporting mathematical problem solving, are unable to be controlled for, thus can be expressed simply as 'mechanism + context = outcome' (Pawson & Tilley, 1997, p. xv). Within the present study, the use of deep researcher involvement with the cases and settings, coupled with observations and interviews (see Chapter 3.4.3), supports a deeper understanding of the causal processes at work. This includes those which may not be directly observable yet may be inferred from behaviours, such as the EF strengths and deficits within each of the cases. The carefully selected questions within the teacher interviews (see chapter 3.4.3) enables a deeper understanding of the EF make-up of each case to be inferred from the qualitative data on pupils' behaviours, through drawing on the previous literature and empirical research. Through observing events and behaviours at the empirical level and analysing the in-depth qualitative data pertaining to each case, the aim is to develop and test theories of the causal mechanisms, which may be operating within the different structures of reality (discussed above) within a system (Maxwell, 2012). Consequently, as with the present study, each case is purposely selected to gain an insight into the specific mechanisms at play within that case (discussed in chapter 3.4.1).

Nevertheless, and perhaps more importantly, justification for using QCA as a research approach and technique should lie within the focus on configurational analysis (the analysis of combinations of different conditions, which give rise to the intended outcome), rather than the number of cases to be analysed (Thiem, 2014). This is due to the underlying assumption within QCA, where causation is considered to be configurational, i.e. based upon specific combinations of conditions, rather than as a direct consequence of one condition in isolation. (Blatter & Haverland, 2012; Rihoux, 2013). This focus on configurational analysis and causation provides the rationale for utilising the methodological approach of QCA for the current study, as the aim is to identify the conditions, or combinations of conditions, along with the underlying mechanisms and contexts, under which the bar model may support autistic pupils with mathematical problem solving.

Based on the discussions of QCA as a research approach thus far, figure 18 positions the current study within this methodological framework, providing an overview of the iterative processes involved within QCA, along with the steps used within the design of the current study.

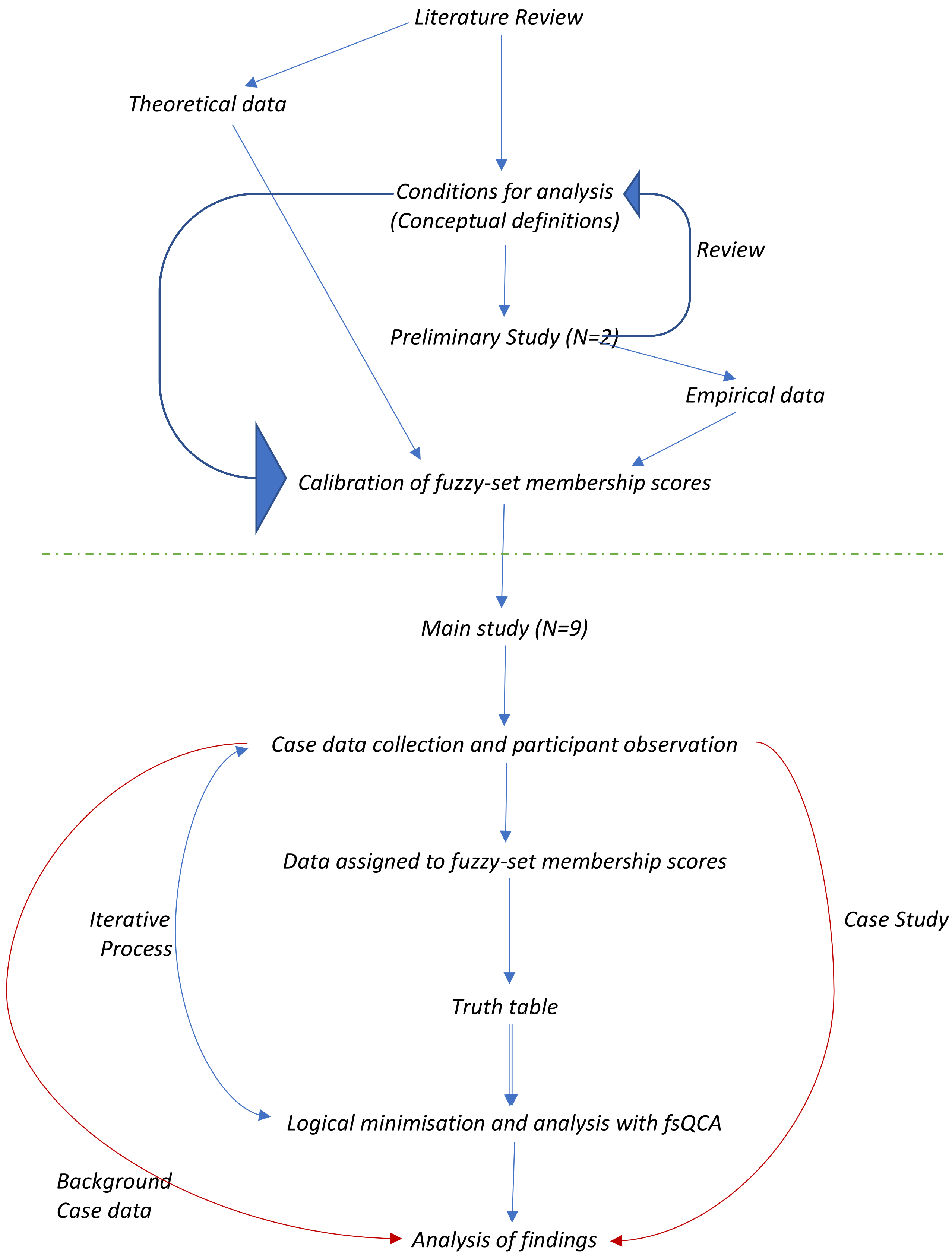


Figure 18: QCA as a methodological approach/research design within the current study

3.3.2 Theoretical assumptions of QCA

QCA can be considered as an umbrella term, as there are several variants on the technique. Such variations are based on the measures used to assign set membership to the conditions being analysed, i.e. the degree to which cases' membership scores belong to a defined set[9]. Where set membership is measured simply as full membership (1), or full non-membership (0), this is referred to as crisp-set QCA (csQCA), as in the example provided earlier. However, in contrast, fuzzy-set QCA (fsQCA), allows for partial set membership, which can be used to express a quantitative difference between qualitatively similar cases, which 'is both conceptually plausible and empirically feasible' (Schneider et al., 2013, p. 326). Unlike in csQCA, where the condition falls into discrete binary (dichotomous) measures - 'the presence or absence of some hypothesised cause and the presence or absence of some outcome' (Eliason & Stryker, 2009, pp. 104–105), the use of fsQCA, as in the present study, is based on 'the degree to which a case belongs to a class defined on some causal condition and a class defined on some outcome' (Eliason & Stryker, 2009, p. 105) (see the Venn diagrams in figures 19 and 20, below). In terms of research interest, fsQCA has attracted the most attention, namely due to its advantage of enabling analysis of conditions based on the 'degree of membership, rather than categorical membership' (as in csQCA) (Roig-Tierno et al., 2017, p. 18). Furthermore, a particular strength of fsQCA, is that it does not fall foul of the major limitation within csQCA to dichotomise the conditions (Rihoux & Ragin, 2009) and consequently lose sight of the differences between qualitatively similar cases. Accordingly, fsQCA accounts for conditions where the measure simply cannot be assigned to a discrete binary category. As with many such conditions within the social world, there are more than two possible categorical measures, which allow for differences between those cases which are similar in a qualitative respect, to be considered (Rihoux & Ragin, 2009; Schneider & Rohlfing, 2016). Despite using fsQCA, any conditions, which allow for a binary measure, i.e. a crisp set (e.g. choice to use the bar model approach or not), may still be (and were) used within the analysis. Due to the conditions being analysed within this study, fsQCA was used, to enable the assignment of appropriate measures to them.

[9] Within set-theoretic methods, the relations between social phenomena are perceived as set relations (Schneider and Wagemann, 2013, p.3).

To enhance the internal validity of the study, the calibration of the fuzzy-set measures, must be transparent and grounded within both theoretical foundations and the case data, which is discussed, in detail, in Chapter 3.5.1. Such calibration enables the conditions to be expressed as partial membership of sets, 'without abandoning the subset relation, which is central to the analysis of causal complexity' (Rihoux & Ragin, 2009, p. 88). Furthermore, in terms of sufficiency, 'if cases sharing several causally relevant conditions uniformly exhibit the same outcome, then these cases constitute a subset of instances of the outcome – and may thus be determined as sufficient for the outcome' (ibid., p.99).

Nevertheless, the use of such quantitative measures to represent the social world faces some criticism. Danermark et al. (2002) remind us that whilst some sociological factors may indeed be measured via an interval scale, other social factors do not necessarily fit this type of measure. Consequently, as discussed later, transparency and clarity over the justification and calibration of such quantitative scales for social factors, must be applied with caution and well-grounded within theory.

In terms of its epistemological underpinnings, QCA is based upon the theory of INUS causation. An INUS condition being a single condition, which is insufficient for producing the outcome on its own, but which is a necessary part of a conjunction (or combination of conditions) which in itself is unnecessary but sufficient for the outcome (Schneider & Wagemann, 2013, p.328). Such INUS conditions can be exemplified through the following theoretical example:

Let A, B, C and D represent conditions, and X represent the outcome.

If the outcome (X) is observed under the following conditions:

Conditions A and B together

Conditions C and D together

The absence of condition C with condition D

Then, condition A is considered an INUS condition. Condition A only gives rise to the observed outcome (X) when combined with condition B, thus it is an insufficient condition on its own. However, it is a necessary condition in the conjunction of A and B together, which this combination. The combination of conditions A and B are sufficient for the outcome (X), however, are unnecessary for the outcome (X), as other combinations can also give rise to X (conditions C and D together and the absence of C with the presence of D).

Therefore, underpinning any QCA analysis is the search for conditions, or combinations of conditions, which give rise to INUS causation for a particular outcome, which the current study seeks to explore.

Underpinning QCA lies three key assumptions: causal asymmetry; equifinality; and conjunctural causation (Thomann & Maggetti, 2017). Causal asymmetry suggests that those conditions, which explain the presence of a particular outcome, may differ from those, which explain its absence. Therefore, in terms of set-theoretical methods, the presence or absence (negation) of a condition giving rise to an outcome, or the presence or absence (negation) of an outcome set, refer to two qualitatively different phenomena. Equifinality assumes that the same outcome may be a result of different, mutually non-exclusive, combinations of conditions (as in the theoretical example above). Finally, conjunctural causation is based on the assumption that single conditions may only result in a particular outcome, if it is operating in conjunction with other, very specific, conditions (as demonstrated with condition A in the example above) (Schneider et al., 2013).

The assumptions discussed above are important to consider within the present study, as the intention is to establish the combinations of specific conditions, under which the bar model may support autistic pupils with successful mathematical word-problem solving. Therefore, it must be considered that any configuration of conditions leading to successful problem solving using the bar model, may constitute one of several pathways. Furthermore, the bar model may only be more beneficial when used in the presence (or indeed absence) of specific conditions. Based on the conjecture of causal asymmetry, it cannot be assumed that the absence of the specific conditions, which give rise to the outcome, will automatically result in the absence of the outcome. Indeed, this may be determined by other specific configurational pathways.

At this point, the alignment between a critical realist philosophy and QCA becomes apparent. Both methodologically, and in terms of analysis, critical realism and QCA have their foundations in causal complexity, theory-laden approaches and limited generalisability (Emmel et al., 2018). Through the application of this view, and with respect to the current research, it can be considered that the ability to solve real-life word problems in mathematics is influenced by a complex set of conditions, as set out in the conceptual framework discussed in chapter 2.3. These conditions by themselves, may be sufficient, or form a necessary part of a more complex conjunction of events and conditions (under specific circumstances), to impact upon an [autistic] individual's ability to solve real-life mathematical word problems.

As an iterative process, 'QCA provides a case-based method for selecting cases for in-depth analysis to supplement its rigorous cross-case analysis' (Cooper & Glaesser, 2018, p. 847). Hence, through the back-and-forth dialogue between the cases and the data, any inconsistencies within the cross-case analysis can be explored in-depth to strengthen the data. The rationale behind this back-and-forth process in analysing such inconsistencies being the context-dependence of the data collected and the influence of the social world on the data, where the value of social and educational research is enhanced by moving beyond that which is simply descriptive (Cooper & Glaesser, 2018). A particular strength of QCA within empirical research, is the focus on 'preserving the intensity of the case-orientated approach, especially its attention to combinations and configurations of causes and conditions' (Ragin, 1994, p. 304). Consequently, given the nature and number of complex frameworks grounded within this study (as seen in the conceptual framework), QCA enables the focus of analysis to be firmly grounded within the complex configurations embedded within these frameworks.

Thus, through its cross-case and within-case analysis, QCA provides an approach to analysing combinations and configurations of conditions, to determine sufficiency and necessity of specific conditions, to provide 'conjunctional causation' (Ragin, 1987), which can support the 'understanding of the mechanisms and processes that explain the links between the conditions (Cooper & Glaesser, 2018, p. 852). Consequently, this type of empirical research, may offer more meaningful and relevant findings to the real-life context of the classroom, as in the case of the current study.

Cooper and Glaesser (2018) argue that QCA enables 'the cross-case component of a mixed methods study to focus more on the holistic case than do conventional quantitative techniques such as regression analysis' (Cooper & Glaesser, 2018, p. 848), which 'average out respective constellations, and often ignore specific, distinct patterns and outliers' (Rihoux & Ragin, 2009, p. 9). Consequently, despite the small-N in the current study (N=9), the use of in-depth data on each of these cases enables detailed analysis between the cases to identify commonalities and anomalies, which in turn may provide an insight into the underlying mechanisms giving rise to the outcome of successful mathematical problem solving.

The use of terminology within QCA is crucial - within this approach, the term 'condition' is synonymous with 'factor' or 'variable', which is more commonly used within other research designs, such as case study. Thus, when analysing the interaction and combinations of conditions within QCA, which ultimately give rise to a specific outcome of interest, the goal is to establish those which are deemed 'necessary' or 'sufficient' (or, in some cases, both) to give rise to such an outcome. Therefore:

- A 'necessary' condition is 'always present when the outcome occurs (i.e. the outcome cannot occur in the absence of the condition)'
- A 'sufficient' condition always occurs when the outcome is present. However, the outcome could also result from other conditions.'
  (Rihoux & Ragin, 2009, p. xix)

Within the context of the present study, the concepts of necessity and sufficiency (discussed earlier) are considered using Venn diagram representations (figures 19 and 20).

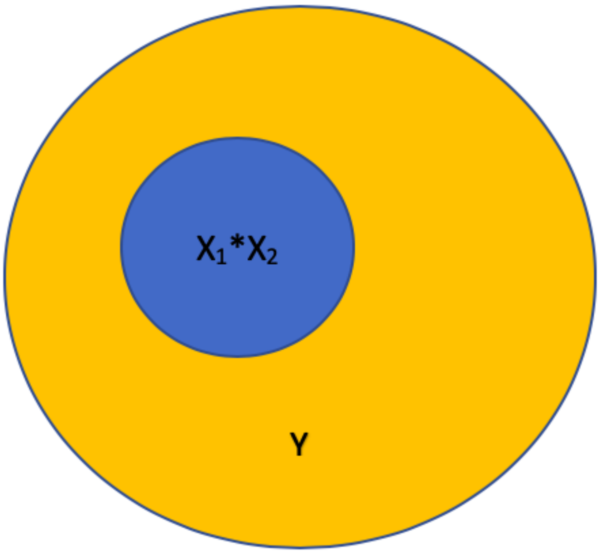


Figure 19: Venn diagram indicating sufficiency. The configuration X_1*X_2 is a subset of outcome Y, therefore indicating a sufficient configuration of conditions for the outcome. As the subset X_1*X_2 does not fill the entire area of outcome Y, other conditions, or configurations of conditions, may be sufficient for outcome Y

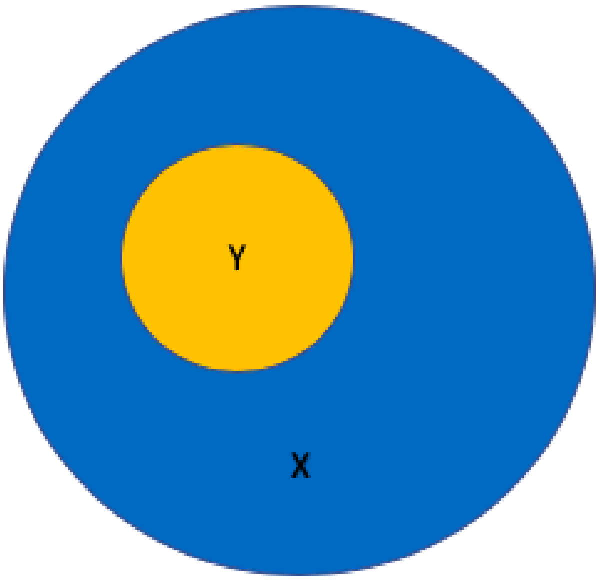


Figure 20: Venn diagram indicating necessity. Condition X is a superset of outcome Y, therefore X is a necessary condition for the outcome Y. In this case, X always needs to be present for the occurrence of outcome Y

Before proceeding, it is important to briefly draw the reader's attention to the formal notation used within QCA, and therefore within the data analysis of this study. According to

Rihoux and Ragin (2009), the formal notation and rules used within QCA, which are 'fixed and stable' (p.14), offer a tool to support replicability and transparency through the iterative nature of the study, adding rigour as a research approach. In terms of conjunctural causation (discussed earlier), or 'multiple conjunctional causation' (Rihoux & Ragin, 2009, p. 8), where multiple pathways of specific combinations of conditions may give rise to a particular outcome, (in which the concept of equifinality (discussed earlier) is also present), notation through Boolean Algebra is used to represent such pathways:

> AB + CD → Y (where the combination of conditions AB ***or***[10] the combination of conditions CD produce the same outcome) (Rihoux & Ragin, 2009, p. 8).

Furthermore, both the presence and absence of conditions, in conjunction with other conditions, may give rise to the same outcome[11]:

AB → Y * aC → Y (where the presence of condition A and condition B together give rise to the outcome (Y), ***and*** the absence of condition A (a) and the presence of condition C together give rise to the outcome (Y)).

Nevertheless, within the ontological foundations of QCA, lies the fact that any form of permanent causality is disregarded due the concept of causality being viewed as complex, and context and conjuncture specific (Atkinson et al., 2020; Ragin, 1987; Rihoux & Ragin, 2009). Furthermore, when considering the irreducibility of ontology to epistemology, as discussed earlier, ontologically, any causal law is considered independent from the pattern of events it may give rise to (Archer et al., 2007). Therefore, within the context of the current study, any findings may be used to infer modest (or 'fuzzy' (Bassey, 2001)) generalisations (discussed earlier), where such findings may be plausibly generalised to those cases sharing the same combinations of conditions, within similar contexts.

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[10] In Boolean Algebra, '+' represents 'or'; '*' represents 'and'.

[11] A lower-case letter represents the absence of a condition, whereas an upper-case letter denotes its presence.

Within QCA, the process of minimisation, which is used to simplify complex expressions often emerging from the data analysis, is based on set theory. Such minimisation aims to 'eliminate redundancies' thus presenting models which are 'causally interpretable in accordance with inus theory' (Thiem, 2016, p. 4). The concept of redundant conditions can be simply exemplified as follows, drawing on the example discussed above:

If the outcome (X) is observed under the following conditions:

Conditions A and B together

Conditions C and D together

The absence of condition C with condition D

Here, we can see that condition C is a redundant condition, as its presence, or absence, in combination with condition D, makes no difference to the outcome. Therefore, in terms of minimisation, we could simplify the above to state that:

The outcome (X) is observed when condition A and B occur together (A*B) or when condition D is present. Thus, in terms of Boolean algebra, following minimisation, the solution formula would be:

$$A*B + D \rightarrow X$$

Due to the configurational approach used within QCA analysis, this allows for combinations of conditions, which are not necessarily present within the cases included within the study, to be considered as counterfactual, using logical remainders. By analysing and representing each of the cases and its associated configurations of conditions, through the construction of a truth table (see Chapter 3.6.2), combined with the process of minimisation (discussed above), it becomes possible to form a logical assumption to the likely outcome of those configurations where there are no cases – hence logical remainders. However, in a move to make such findings more reliable, the use of consistency measures (discussed in Chapters 3.6.2) within QCA analysis has developed since Ragin's (1987) initial use of QCA (Rihoux, 2013; Schneider & Wagemann, 2013). Such measures of consistency, indicate the degree as to how closely the pattern of cases within the study, matches the solution outcome; in more technical terms – 'the percentage of causal configurations of similar composition which result in the same outcome value' (Roig-Tierno et al., 2017, p. 17). Coupled with this, is the emphasis on engaging with theory when attempting to analyse such logical remainders (through the conceptual framework within the current study), in an attempt to increase the overall validity of the findings and to address the often critiqued issue of 'limited diversity' (where there is insufficient empirical evidence to support all of the logically possible combinations of conditions) within QCA (Rihoux, 2013, p. 240). Thus, the engagement with current theory through the conceptual framework underpinning this study, defends the use of a small-N.

Despite the use of logical remainders drawing criticism from some researchers (Baumgartner & Thiem, 2017) due to the lack of empirical evidence on which these are based, Rihoux and Ragin (2009) defend their use, arguing that 'their inclusion does not change anything about the properties of the empirical (observed) cases', and furthermore, enables the researcher to move 'towards theoretical elaboration' (p.153), therefore supporting the rationale for a small-N within the current study. They continue this argument by suggesting that, as with all scientific or social research disciplines, any form of generalisations made from the data, 'necessitates going beyond the observed cases' (p.153). As such, in line with the disregard for permanent causality (discussed above), within QCA, such generalisations may indeed go beyond the observed cases, however, are limited to those sharing the same configurations of conditions as the observed cases. Therefore, any findings from the current study, although not generalisable to the whole autistic population, may be generalised to those individuals who share the same conditions, and are contextually similar, to the cases within the current study.

3.4 Data collection

The following sections discuss the sampling techniques used within the current study, along with the selection and de-selection criteria. Arguments are presented for the use of a relatively small-N in the current study. The selection for conditions for analysis are discussed and the rationale provided, before moving on to discuss the data collection methods applied.

3.4.1 Case selection and sampling

QCA, it is best suited to purposively selected samples and explorative research designs (Rihoux & Ragin, 2009; Thomann & Maggetti, 2017) due to the focus on the interactions of specific conditions and outcomes of interest, as in the present study. Hence, case-selection must be considered carefully, to ensure such specific conditions of interest are present (or absent) in the selected cases (the conditions for analysis are discussed in detail in section 3.4.2).

Of significance to the present study, QCA is well suited to studies involving a small number of cases (N), where N = 2-15, which is often an insufficient number of cases to perform large-scale statistical analyses (or even small-scale statistics with confident reliability), but too many to enable a full, in-depth case study on all cases (Jopke & Gerrits, 2019; Thiem, 2014). Within the current study, the rationale for the relatively small number of cases (N=9) is based on the following justifications, which are further discussed throughout this section:

- The intention to carry out in-depth analysis on the case data provided for each of the participants, in order to generate a rich data set, which supports the analysis of conditions within QCA;
- The guidance provided by researchers in the field of QCA (Jopke & Gerrits, 2019; Thiem, 2014) regarding optimum sample sizes for QCA studies;
- Challenges in sourcing and accessing cases, which fully meet the inclusion criteria set out within the study (discussed below);
- Ensuring specific case contexts and conditions, along with clear boundaries for the case to support replicability of the study and to add rigour to the methodological approach of QCA;
- To limit the influence of wider conditions, not analysed within the current research, on the findings from the data, in order to increase the robustness of the study;

Within QCA, ‘each case is considered as a whole (holistic approach), and the effects of variables are assessed in the context of the case. Cases are therefore represented as configurations of variables' (Rihoux, 2013, p. 234). Consequently, an important aspect of case selection within QCA is the use of theory-driven selection, to include 'both the important or typical cases and the more paradoxical or contrary ones' (Rihoux & Ragin, 2009, p. 7). As the aim of QCA, is to establish causal pathways, consisting of combinations of specific conditions giving rise to a particular outcome (CMO configurations as discussed earlier), the final number of cases (pupils) within the study, may be determined by the range of configurational pathways of conditions observed within these cases. Due to the iterative nature of QCA, re-specification[12] of cases, or recalibration, may be required in order to reduce any skewed data from the analysis (Thomann & Maggetti, 2017). Furthermore, the cases selected, 'must share enough background characteristics which in turn can be considered as 'constants' within the analysis' (Rihoux & Ragin, 2009, p. 20). Throughout the study, cases may be added, or dropped, to maximise the analysis on the specific configurations of interest. This further supports the limited sample size used within the current study, in order to enable the process of additional cases to be added (or, indeed dropped) where the analysis of data indicates configurations of interest, which may not be supported by empirical data (i.e. logical remainders, as discussed earlier). Similarly, in the instances where cases included within the analysis suggest inconclusive findings, the addition of cases sharing the same conditions (or the conditions of interest generated from the data) can be added throughout the research process, to increase the validity of the findings. Should specific cases generate ambiguous findings, these cases may be dropped (although not discounted) from the analysis to ascertain any anomalies or to verify the regularities found within the configurations analysed. To ensure rigour and robustness within this iterative process, all decisions regarding the addition or removal of cases within the analysis should be reported clearly.

Crucial to the current study, is the understanding of what is meant by the term 'case'. Here, the term 'case' refers to each pupil within the study, and the associated combination of conditions present (or absent) in the case. In terms of the case boundaries, the cases in the

[12] The re-specification of cases within QCA refers to the flexibility to add or remove cases to the study, based on the analysis of configurational pathways. Such re-specification enables the analysis to retain its focus on the conditions of interest, and to therefore disregard those cases, which are deemed irrelevant to the study, or seek further examples of cases containing the conditions of interest.

current study are bounded by their combinations of conditions and the context within which the cases reside (the individual schools). This is of particular significance for the entire study, as significant background data is collected and analysed for each case (pupil) and the combinations of conditions, coupled with the contextual factors of the setting in which they reside, provide key insights into the underlying mechanisms from both cognitive and social perspectives (discussed further within section 3.4.3). Consequently, along with the behavioural- and academic-specific details of the cases, qualitative data pertaining to the application of the bar model within each setting, is used to develop the context-specific characteristics in which each case is bound. Within the current study, cases include both those pupils with a formal diagnosis of autism (N=7) as well as a small number of neurotypical pupils (N=2). However, those cases where autism is present are the cases of interest, with regards to the in-depth data collection and analysis, as the research questions focus on the exploration of any necessary or sufficient conditions specifically associated with autistic pupils. However, the use of the neurotypical cases is used to act as a control measure within the current study, to ascertain whether the findings are specific only to those autistic cases, or more widespread across the population. Within case-based QCA, each case is considered not only as a configuration of the conditions studied and the contextual factors associated with the case, but the outcome condition of interest should be clearly defined and justified (Jopke & Gerrits, 2019; Rihoux & Ragin, 2009; Schneider & Wagemann, 2010), which in the current study is the successful solving of mathematical word problems. Following this key principle, the outcome of interest for the current study, based on the theoretical findings from the literature review, is defined as:

> *An individual's ability to make a choice of strategy/visual representation, and execute the strategy successfully, to solve a 2-step, mathematical word problem. The outcome is measured using the Visual Representation Observation Form (Adapted from* (Bae, 2013)*) and reaching the correct solution to the problem. The outcome is calibrated to a fuzzy-set scale, based on theory and preliminary case data analysis (see calibration of conditions in section 3.5.1).*

Using a small, purposive sample, rather than random sampling techniques typical of quantitative studies, is based on identifying those subjects who best exhibit the phenomena of interest. Furthermore, it is based on theoretical criteria to determine the cases' relevance to the research question; in this case, pupils with autism within primary school mathematics lessons (Maxwell, 2012; Thomann & Maggetti, 2017). The outcome of this study (described above), as with most qualitative studies, is to 'understand the meanings, processes and local contextual influences [on mathematical problem solving ability] for the specific settings or individuals [autistic pupils] studied' (Maxwell, 2012). However, the rationale and justification behind case selection must be transparent in order to strengthen the internal validity of QCA (Thomann & Maggetti, 2017, p. 17).

Due to the relatively small N, used within the present study (N=9), it is suggested that in order to 'narrow down the 'conditions of occurrence' for exploratory purposes, to identify some factors [conditions] that may possibly be responsible for the respective outcome', the study should be based on the most similar, different outcome (MSDO) design (Rihoux & Ragin, 2009, p. 22). Further supporting the use of a small-N within the current study, in order to generate diversity amongst the cases analysed, it is recommended to ensure a 'maximum heterogeneity over a minimum number of cases' (Rihoux & Ragin, 2009, p. 21). Complementary to this idea, is that the selection of cases should be homogeneous, in terms of background characteristics, yet heterogeneous, in terms of conditions and the outcome, thus enhancing the quality of the data produced, in addition to supporting the concept of causal asymmetry (discussed earlier) (Hirzalla, 2020). Furthermore, within QCA, it is advantageous to include cases which include both positive and negative outcomes (ibid.), hence the re-specification of cases, as discussed earlier. This is executed through the specific inclusion (and exclusion) criteria used within the current study for the selection of cases (discussed below).

An advantage of using a small N, according to Rihoux and Ragin (2009) is that familiarity with the cases is strengthened, therefore mitigating both measurement and coding errors, when determining the membership thresholds for each combination of conditions, thus increasing the validity and robustness of the findings. Furthermore, the use of a small-N allows for the researcher to gain, and base the interpretation on, an in-depth, complex case knowledge (Hirzalla, 2020). As a consequence of the iterative nature of QCA as a process, the specific number of cases to be analysed within the study is often difficult to establish a priori (Rihoux & Ragin, 2009). In support of this, Thiem (2014) argues that 'if researchers only explore simple conditions a priori, [most] data-fitting conditions will be missed' (p.495). Hence the sometimes-applied iterative process of addition and removal of cases, allowing for engagement with the case data to continue to drive the selection of conditions for analysis within QCA. The theoretical suggestions discussed above are used to guide the case selection criteria for the current study.

Based on the review of literature carried out within this study, the table below uses PICO (population of interest; phenomena of interest; and context) to assign the criteria for case selection in the current study. The transparent use of such selection criteria is used to strengthen the validity of the study (Legewie, 2013; Rihoux & Ragin, 2009; Schneider & Wagemann, 2010; Schneider et al., 2013).

	Inclusion	**Exclusion**
Population of interest	• Primary or elementary age • Pupils in Year2-Year 6 • Diagnosis of ASD • Verbal • Male	• Age <6 • Co-morbid conditions (relates to EF deficits associated with co-morbid conditions from literature review)
phenomena of Interest	• Mathematical problem solving • Word problems • Visual representations • Use bar model >1 year	• New to SBM or using for <1 year
Context	• Mainstream education • England	• Special education • Not primary education

Table 4: The case selection criteria, theoretically driven and presented through PICO criteria

In total, nine schools, which were geographically spread throughout England to avoid any local or regional policy influences, were contacted to take part in the study. Of those nine, five of the schools responded and agreed to participate within the research. Later, each of the participating schools was contacted to source a group of neurotypical pupils, enabling any findings relating to autistic cases to be considered against a wider population, to explore whether such findings were indeed common amongst, or differing to, neurotypical cases. The criteria for selection of this group of cases was identical to the original group, except for a diagnosis of autism (or any other condition). The same background data on these cases, along with the same mathematical task and discussion were carried out with this group of pupils. Due to the difference in diagnosis (i.e. no diagnosis), an ethical amendment was sought and granted for this aspect of the study (see Chapter 3.7).

Within these five schools, a total of 15 pupils (n=11 with a formal diagnosis of autism and n=4 neurotypical) were identified for the study. The rationale for identification, selection and deselection of cases is discussed in detail above. The identification of the pupils within each school was made by either the head teacher or the SENCo, in line with the inclusion criteria initially sent out to each school. Two cases (neurotypical) failed to receive parental consent for the study and were therefore excluded. Upon data collection, a further four cases (ASD) were excluded from the study due to not meeting the inclusion criteria stated within the research: two pupils from one school, where it emerged that the bar model had only been introduced within the last six months (SchF); and two pupils from another school, who were non-verbal (SchA). Consequently, the final number of cases used within the data analysis was N=9 (n=7 ASD and n=2 neurotypical), from four geographically spread schools.

Figure 21 below, documents the final participating schools and cases used within the final study.

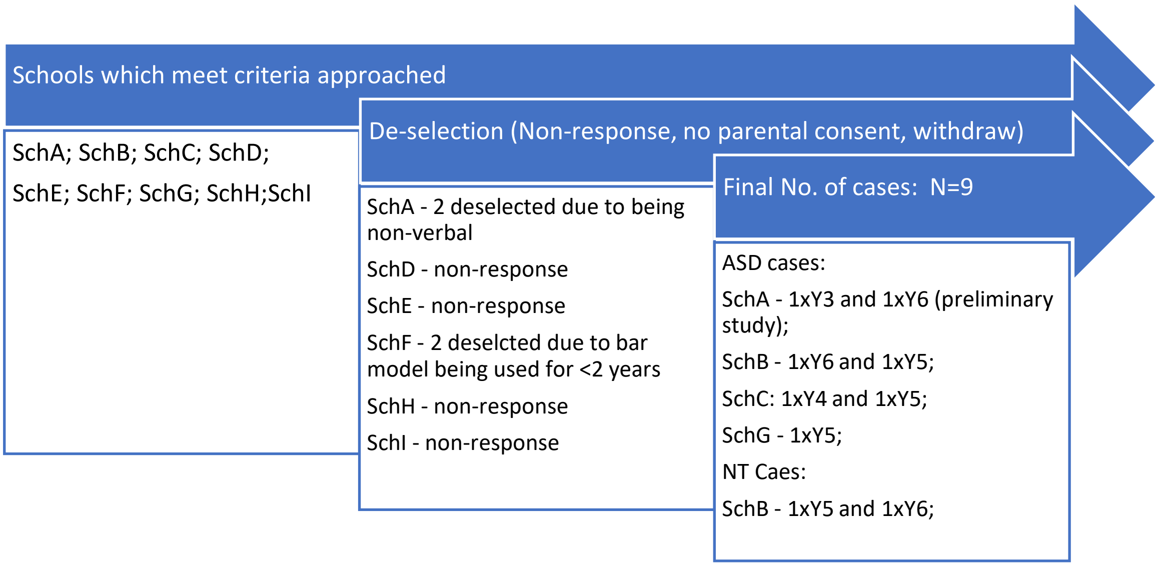


Figure 21: Selection and de-selection of schools and cases for the current study

Throughout the data collection, identification codes were applied both to the schools and the pupils (cases) within the schools. The codes used were as follows:

- Each school was allocated a letter code: SchA; SchB; SchC; and SchG; (Note: SchD, SchE and SchF do not exist, as these schools either provided a non-response to the invitation to participate within the study or were de-selected as discussed above);
- Each pupil (case) was allocated a number, based on the number of cases within each school. This pupil (case) number was then attributed to the school code: e.g. four cases were based in school A (prior to deselection), hence coded as: SchAP1; SchAP2; SchAP3 and SchAP4. On the other hand, only two cases were attributed to school C, hence the codes: SchCP1 and SchCP2; and so on.
- Each class teacher was allocated a code, comprising the school code, plus the teacher number within the school. E.g. two different class teachers were involved in the data collection in SchC (one for each of the pupils) therefore the codes SchCCT1 and SchCCT2, and so on.

Now that the selection of cases has been discussed, the following section moves on to discuss the selection and rationale for the conditions for analysis within the current study.

3.4.2 Selection of conditions

As with case selection, within QCA, the selection of conditions for analysis should be theory driven (Jopke & Gerrits, 2019; Rihoux & Ragin, 2009). The use of existing theories and hypotheses, as an explanatory framework, which are theory-driven, can be used to eliminate some of the many potentially less relevant conditions (which are inevitable within the context of social science) for analysis a priori. Nevertheless, in a similar manner to the iterative process used to add or drop cases throughout the study, based on confirmation or falsification of hypotheses, the conditions for analysis can be amended throughout the process, based on preliminary analytical findings. Whilst this often presents as another significant area for critique amongst some researchers (Baumgartner & Thiem, 2017), Rihoux and Ragin (2009), whilst not disputing this critique, offer suggestions of good practice within QCA as a means to restrict such criticism. They suggest that to reduce this issue, researchers should try to maintain homogeneity amongst the cases, in terms of the background conditions, whilst 'increasing their diversity with regards to the combinations of values of the conditions' (p.157), as discussed above within the current study.Consequently, the conditions should be, to a certain extent, considered at an abstract level, as within QCA, they are used within cross-case analysis, as opposed to relating directly to the specific context of each of the cases (Jopke & Gerrits, 2019).

In support of the iterative nature of the QCA, and the need to ground the selection of conditions for analysis on both theory and data, the number of conditions for analysis can be kept to a minimum, hence 'the fewer number of 'causes' we need to explain a phenomenon of interest, the closer we come to the 'core' elements of causal mechanisms' (Rihoux & Ragin, 2009, p. 27). This is an important consideration within the QCA process, particularly when working with a small N as within the current study, as 'the number of possible logical combinations of conditions can quickly exceed the number of cases, and the empirically observed cases will occupy only a tiny proportion of the 'logical space'' (Rihoux & Ragin, 2009, p. 27), i.e. the resultant logical remainders, as discussed earlier. One consequence of this is the difficulty in carrying out the minimisation process (Hirzalla, 2020). This can be seen by considering the exponential growth of possible combinations (where crisp-set, binary categories are used), as the number of conditions is increased:

1 condition = 2 possible combinations (2^1)

2 conditions = 4 possible combinations (2^2)

3 conditions = 8 possible combinations (2^3)

6 conditions = 64 possible combinations (2^6)

…and so on.

Ideally, it is suggested that the number of cases used, should be four times greater than the number of conditions (Hirzalla, 2020). However, as is the case with the current study (where N is small, but the number of possible conditions for analysis is large), this is not always feasible in reality. Where this is the case, the number of conditions should be kept to the minimum possible (ibid), to increase the reliability of the findings.

When we begin to consider fsQCA, the number of possible combinations of pathways, is increased even further, due to the move beyond binary categories for assigning the membership of cases. However, in support of the continued dialogue with the cases, and the use of initial data analysis to drive the refinement of the conditions for analysis, 'a single macrovariable might be used to replace [multiple] substitutable causal conditions joined together by logical ***or***, which dictates using their maximum membership score' (Ragin, 2005, p. 14).

In terms of the selection of conditions within the current study, the significance of critical realism, along with the existing theories within the literature and previous empirical studies, is paramount. In line with CR, an individual's ideas and feelings are viewed as equally as real as physical, observable objects and processes. Based on this belief, and using the literature to guide the conditions for analysis within the current study (Cooper et al., 2018; Gravemeijer, 2020; McLeod, 1985; Muis et al., 2015; Schoenfeld, 1985; Silver, 1985), pupils' self-perception of their own mathematical ability is used as one of the key conditions for analysis as this concept is considered to be a real phenomenon in line with CR research.

In terms of reading ability, the studies discussed in the literature review (Bae et al., 2015; Björn et al., 2016; Boonen et al., 2013; Jones et al., 2009; Oswald et al., 2016; Özsoy, 2015;

Wei et al., 2015; Whitby & Mancil, 2009) support the decision to read aloud the word problems to the participants within the current study, as although reading aloud may eradicate any impact of reading fluency, the skills of text comprehension are still required for the participants to understand the problems. This decision within the current study enables the use of reading, as a condition, to be focused upon comprehension skills, rather than word decoding and fluency skills. This draws upon on the research discussed above, where it is suggested that both reading comprehension and mathematical word problem solving may both be underpinned by the same EF skills (Björn et al., 2016; Özsoy, 2015). Consequently, the decision enables the current study to focus on the potential underlying mechanisms of the EFs within these two processes.

As discussed in the literature review, pupils' mathematical attainment is considered significant in terms of their overall problem solving skills (Cooper et al., 2018; Hord et al., 2016; Rau, 2017), therefore becoming a condition for analysis within the current study.

Therefore, based on the above considerations and the literature review, the conditions for analysis within the cases of the preliminary study of the present research, are as follows:

- Pupils' reading comprehension;
- Pupil's current level of mathematical attainment;
- Choose to use the bar model to solve mathematical word problems;
- Visual representation application and accuracy;
- Pupil's self-perception of mathematical problem-solving ability.

Following on from providing a discussion and rationale behind the case- and condition-selection for the current study, the specific research instruments, used in the data collection, are now discussed in the following section.

3.4.3 Instruments for data collection

This section discusses, in detail, the methods of data collection (research instruments) used within the current study, driven by the philosophical framework of critical realism discussed earlier. As the overall research design utilises QCA, both as a research approach and technique, this provides the overarching theme of the chapter, whilst considering, in turn, the specific methodological tools for data collection.

At the empirical level within social science research, Fletcher (2017) suggests two types of data collection which can take place: extensive, for example the use of statistical data; and intensive, such as in-depth interpretive data collected from interviews and observations. Within the current study, the primary focus is on intensive data, collected and analysed through QCA, which combines both qualitative and quantitative forms of data analysis (Cooper & Glaesser, 2018). Rather than relying on the discrete fields of quantitative and qualitative data, QCA allows the details of individual cases, which may easily get lost during statistical aggregation, to be maintained (Cooper & Glaesser, 2018). This argument builds upon Rihoux and Ragin's (2009) defence of QCA as a research technique, in that within the 'statistical paradigm, the various explanatory variables must be considered as statistically independent' (p.148), whereas this is not the case within QCA. Furthermore, QCA allows the researcher to use a systematic approach to assess causation through cross-case comparison, and within-case complexity (Hirzalla, 2020).

The main research instruments used within this study are semi-structured interviews and observation and discussion with pupils during a mathematical problem-solving task, each discussed briefly below.

3.4.4 Semi-structured interviews

One of the research instruments utilised within the current study is that of semi-structured interviews. The semi-structured interview is used to gain in-depth, background case-based information on each of the pupils, gained from the teachers' perspectives. A key consideration in conducting any type of interview is the 'framing of real-life events' (Denzin

& Lincoln, 1994, p. 370), which incorporates the type of interview conducted as well as the capturing and analysis of the data. Furthermore, the interview provides some insight into the extent to which the bar model, as a visual representation, is embedded within mathematics teaching for each of the cases.

The interview is a powerful tool for producing (or constructing) knowledge through social interaction and communication (Kvale, 2007). However, it should be considered, as discussed earlier, that such knowledge cannot be reduced to reality and that the responses given within the interview, are likely based on empirical observations rather than the actual or real events underlying such observations. Furthermore, the social relationship established between both the interviewer and the interviewee is crucial in determining the accuracy of the knowledge produced (ibid). Such social relationships may impact upon the internal validity of the interview responses (discussed later), particularly in terms of the respondent providing information deemed to be that which the interviewer wishes to hear.

An advantage of semi-structured interviews, as opposed to more formal, structured interviews, is that it allows for the use of probing questions and prompts (Denzin & Lincoln, 1994). This, in turn, enables the researcher to use a funnelling approach to gain deeper insight into the teacher's perspectives and understanding of the pupil's background, in relation to the key conditions for analysis within this study. In addition, the flexibility of the semi structured interview enables unexpected responses to be followed up, along with interpreting questions to clarify and ensure reflexivity during the co-construction of the interpretations of the teachers' responses. Whilst a more structured interview may indeed enhance the reliability of the study (Silverman, 2010), the imposition of such structure does not allow for the flexibility in questioning through probing and checking, which can enhance the internal validity of the interview.

From a realist perspective, the aims of data collection are to explore the potential causal processes, which are in operation between the individual pupils and the outcome of successful mathematical problem solving (Emmel et al., 2018). The semi-structured interview schedule used within this study (see appendix i), allows for some specific quantitative measures (e.g. chronological age, level of reading and mathematical attainment) to be collected on the pupils, as well as some qualitative background data based on the teacher's perspectives (e.g. specific areas of difficulty). The schedule design is based upon the findings from the literature review (discussed earlier), considered to explore the key areas of potential influence on autistic pupils' mathematical problem-solving ability.

The data collected through these semi-structured interviews with class teachers, was recorded as notes, directly onto the interview/discussion prompt, as can be seen in appendix ii. The decision was made not to voice record these interviews, as the first interview conducted took place whilst the teacher was supervising a class, hence significant background noise. In order to maintain consistency between cases, the same approach was continued throughout. However, to enhance the validity of the data, the class teachers were asked to verify and validate the notes made for accuracy. These notes, in conjunction with the participant observation and discussion data (discussed later) provides background, in-depth case-based data on the pupils within the study, which is then be referred to throughout the iterative data analysis process. The data collected on measures such as prior mathematical attainment and areas of strength and weakness of the pupils, was then used to inform and strengthen the calibration process (see section 3.5.1) of the conditions for analysis.

A common criticism of any type of interview are the threats to validity due to interviewer bias. Such threats may result from preconceived attitudes of the researcher, misunderstandings or misinterpretations and the tendency for the interviewer to seek answers to support their own theories (Cohen et al., 2018). However, according to Denzin and Lincoln (1994), three other common sources of error, which can reduce the validity within interviews are:

1. The respondent providing answers deemed 'desirable' to the researcher;
2. The wording of the questions used within the interview (lack of clarity or leading questions);
3. Flawed questioning techniques of the researcher.

(Denzin & Lincoln, 1994, p. 364)

In relation to the second point raised by Denzin and Lincoln (1994), is also the threat imposed by whether the questions asked, actually measure what it is they claim to be measuring (Cohen et al., 2018). From a critical realist perspective, this is significant as language is paramount in constructing individual's conceptual frameworks associated with terms and therefore 'meaning' associated with concepts is inherently value-laden and is 'never fixed' (Danermark et al., 2002, p. 27), or as Kvale (2007) suggests are representative of their 'lived world' (p.19).

Nevertheless, through the use of reflexivity throughout the interview process in the current study: clarifying questions, detailed field notes and respondent validation, the use of simple language and the partially fixed schedule used, such threats are reduced (Cohen et al., 2018; Denzin & Lincoln, 1994). Kvale (2007) suggests that such clarification of meaning within qualitative interviews 'corresponds to exactness in quantitative measurements' (p.12). Furthermore, the use of such clarification, supports 'self-understanding', significant to establishing the understanding of meaning, which in turn, enhances the 'communicative validity' (ibid. p.125) within the interview.

However, in contrast to Denzin and Lincoln's (1994) recommendations, the use of inter-subjective agreement (which shares some of the same principles as respondent validation) as a means of confirming validity, does not fully align with Bhaskar's (2007) critical realist divergence from relativism (Groff, 2004). Despite this more relativist approach, respondent validation was used as a validity enhancing tool within the current study.

Whilst the generalisability of interview findings is often criticised, due to the subjectivity of the responses, Kvale (2007) suggests that rather than generalisations per se, we instead consider 'transferability of knowledge' (p.87), where contextual and social factors are accounted for. Within this study, Kvale's (2007) suggestion provides a useful alternative to generalisation, as the focus is on developing in-depth case-based knowledge of the pupils within the study, where specific contextual factors are indeed relevant to the analysis of combinations of conditions giving rise to (or not) particular outcomes within mathematical problem solving.

3.4.5 Mathematical problem-solving task

The mathematical problem-solving task was used to ascertain the key outcome measure for the current study: reaching the correct solution to a two-step, mathematical word problem, matched to pupils' age through National Curriculum objectives. Each of these problems were designed to lend themselves to being solved using the bar model.

In addition to the use of voice recordings for the discussion with pupils, detailed field notes were also kept from the observation, in order to enable non-verbal cues (such as facial expressions and body language) to be captured, as these cues supported the interpretation and meaning of the data collected (Kvale, 2007).

The use of multimethod (or complementary) approaches to data collection enables the production of more extensive results, through which triangulation of key findings can be utilised to increase the internal validity and consistency of the research (Denzin & Lincoln, 1994). Within the present study, the use of a mathematical problem-solving task, which incorporates aspects of participant observation and semi-structured interviewing with the pupils, was used to complement the semi-structured interviews with the teachers.

In order to establish the decision-making processes and identify the strategies used by pupils to solve mathematical word problems, hence 'actively witnessing the phenomena' (Denzin & Lincoln, 1994, p. 378), a short task, delivered in conjunction with observation of, and discussion with, the pupil was utilised. Whilst some aspects of this task draw on participant observation, which is rooted within symbolic interactionism, as well as a structured observation schedule (see appendix iii), as a data collection method, there are also elements of interview and discussion used to generate a clear understanding of the pupil's approaches and reasoning to solving mathematical word problems.

Whereas Bae's (2013) study involved one-step word problems as a measure of problem solving ability, Björn et al. 's (2016) study utilised two-step word problems, where the interrelation between skills in text comprehension and mathematical word problem solving ability appear to be more pronounced (Björn et al., 2016). Consequently, the current study uses two-step word problems to enable the potential influence of reading comprehension ability on mathematical problem-solving ability to be explored within the sample. This decision, coupled with the decision to read aloud the word problem to the participants, further supports the potential to explore any mediating influence of the EFs, as mechanisms underpinning these processes.

A mathematical word problem was presented to the pupils (matched to pupils' age and aligning with National Curriculum expectations (DfE & DfE, 2013; NCETM, 2018))(see appendices iv-vii), all of which are possible, and lend themselves to being solved using the bar model as a visual representation. Below is one example of such a question, based upon the Year 5 National Curriculum expectations (DfE, 2013):

Sharon and Tim each had a collection of football stickers. Tim had 5 times as many as Sharon. He had 150. How many did they have altogether? (*Y5: solve problems involving multiplication and division, including scaling by simple fractions and problems involving simple rates.)*

Figure 22: An example of a word problem, which can be solved using the bar model, based on the Year 5 National Curriculum expectations

The decision was made to only administer one-word problem to each student based on recommendations made by the class teachers during the initial contact discussions. The rationale for this was the threat to pupils' level of concentration and focus during the data collection process. To maximise the quality and depth of the data collected from the discussions with pupils, the decision to complete only one mathematical problem-solving task was made. Although the completion of a greater number of tasks would have provided more data on the outcome measure, the researcher felt that the benefits of a rich qualitative data set from the discussion of the pupils, outweighed this risk.

A set of predetermined discussion prompts (appendix iii) were used both before and after pupils' completion of the task to provide case-based data regarding pupil self-perception of mathematical ability, along with a rationale for the decisions and strategies made during the problem-solving process. One key decision to be observed here, is that of whether the pupil chose to use the bar model, or indeed, selected an alternative strategy or representation – a significant factor for analysis within this study. Here, the possibility of 'negative cases' (in terms of those pupils not choosing to use the bar model) being introduced into the study is highly likely, and indeed purposeful. In addition to the concept of negation (discussed earlier), the search for such 'negative cases' enhances the external validity of the findings (Denzin & Lincoln, 1994, p. 381).

When considering the pupils' accuracy, in terms of representation and magnitude, when using visual representations in mathematical problem solving, the study draws upon the visual representation observation form (VROF), adapted from Bae's (2013) study (see appendix viii). This schedule allows the pupils' accuracy of representation, along with considering whether the correct solution was reached, to be measured.

The use of the preliminary study (see Chapter 3.5) within the current research, offers a second step to enhancing the internal validity, through refinement of the observation categories used within the main study (Cohen et al., 2018). Another significant threat to the internal validity of such a method of data collection is that of inter-rater reliability. To reduce the impact of this threat within the current study, and due to the vulnerability of the pupils, an additional adult was present during all mathematical problem-solving tasks to ensure familiarity to the pupil, but to also act as an additional check for understanding and interpretation of the pupils' responses. In all cases, the additional adult was familiar to the child (either the teacher or teaching assistant) to reduce any anxiety of introducing another unknown adult to these vulnerable pupils (discussed in chapter 3.7).

As with the design of the semi-structured interview, the discussion prompts used within the mathematical problem-solving task, are based on findings from the literature (see literature review). These findings enable the identification of potential key conditions influencing the problem-solving ability of autistic pupils within the current study.

However, as with all data collection methods, the approach of observation and discussion used within the administration of these tasks, is open to threats from both external and internal validity. To reduce such validity threats, rigorous checks were put in place to minimise these. Threats to external validity, such as how representative the participant is of the population of interest, can be reduced through the clarity of the sampling process and case selection rationale used (see section 3.4.1). Furthermore, an additional threat to the validity of this method may include the researcher's unawareness of the participants' prior events (Cohen et al., 2018). However, this threat to validity was minimised using a semi-structured interview with the class teacher to gain in-depth knowledge, held prior to the observation of the mathematical task.

Stem sentence completion task

One test for assessing pupils' central coherence (discussed in chapter 2.1), is the stem sentence completion task (Booth & Happé, 2010). This task, which has 'proved to be sensitive to individual differences independent of IQ among typically developing young people' (Booth & Happé, 2010, p. 389), requires pupils to complete a given sentence stem with a response as quickly as possible. The response given, may be indicative of local or global processing of information by the pupils, and hence provides a useful indicator of weak central coherence. The example below indicates potential local and global responses to a stem sentence used within the current study:

Stem sentence: *In the sea, there are fish and…*
Example of a local response: chips
Example of a global response: sharks, whales
(Booth & Happé, 2010)

Completion of the sentence with a local response, is suggested to be indicative of local processing, and hence weak central coherence. Findings from previous studies by Booth and Happe (2010) suggest that this test can identify WCC amongst autistic individuals and that 'the tendency to provide local completions is independent of intellectual ability' (ibid. p.389-390). Furthermore, the authors argue that this test addresses 'individual differences in cognitive style, rather than just ability' (pp.389-390). Consequently, this task was added to the data collection.

A full breakdown of the stem sentences used with the pupils, along with likely local and global responses, can be seen in appendix ix. Completion of the task is scored using a 3-point system:

For every local completion, a score of 0 is recorded;
For every global completion, provided within ten seconds, a score of 2 is recorded;
For a global completion, when the response is delayed by more than ten seconds, a score of 1 is recorded.
(Booth & Happé, 2010, p. 382)

The test contains ten stem sentences, enabling a maximum score of 20 to be achieved. Filler sentences (those which would not usually provoke a different local or global response) are interspersed between the stem sentences. One such example of a filler sentence used within the current study, is as follows:

> *I was given a pen and…* (Booth & Happé, 2010)

Within the study carried out by Booth and Happe (2010), a '95% agreement was reached between the test administrator and an independent coder (who was blind to the group membership of the participants)' (p.382), thus enhancing the reliability of this test.

3.5 Preliminary study and initial calibration of conditions

This section discusses the rationale, data collection and analysis of the preliminary study, which is positioned within the philosophical and methodological frameworks discussed above. The purpose of the preliminary study is to enable the initial data collected to be used for the calibration and recalibration of the conditions for analysis. Unlike a pilot study, which may only set out to test the research tools, the data from the preliminary study is used beyond this to check the calibration and guide any recalibration of the condition measures. Through transparent recording of the preliminary data collection and analysis, the calibration and recalibration process for the conditions for analysis becomes replicable and hence enhance the internal validity of the overall study (Rihoux & Ragin, 2009; Schneider &

Wagemann, 2010). Furthermore, as the data from the preliminary study forms the basis of the calibration and recalibration of conditions for analysis, these data are also included within the main study. Through the refinement of these measures (see recalibration of conditions in 3.5.2), the internal validity of the research is enhanced, and transparency of the research instruments is demonstrated.

The school used within the preliminary study (SchA) was a larger than average sized primary school, with 441 pupils on roll, located within a suburban West-London borough. The school has an above average proportion of pupils from several groups: minority ethnic backgrounds; pupils eligible for free school meals; and pupils identified with special educational needs (SchA Ofsted report (2014)). Within the school, numeracy skills are strong from the early years and strong progress is evident, particularly with disadvantaged students (SchA short Ofsted inspection (2018)).

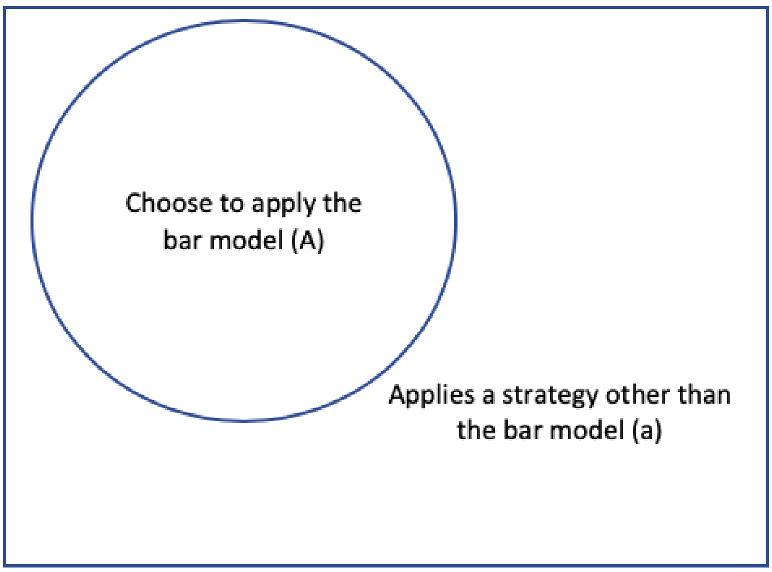
Data for the preliminary study was collected from two pupils, using semi-structured interviews with the pupils' class teachers, in order to provide background case data for each pupil, and a discussion with the pupils around completion of a two-step word problem, using the research instruments discussed above. The semi-structured interviews provided background case data on each of the participants, such as their current level of mathematical attainment and the areas of strength and difficulties experienced by the individuals both in mathematics and wider aspects of the curriculum and school life, including potential social factors. Question prompts for the interview schedule were derived from the literature relating to key factors considered important in mathematical problem solving (see appendix i) (Björn et al., 2016; Boonen et al., 2013; Cooper et al., 2018; Kleinert et al., 2015; McLeod, 1985; Morin et al., 2017; Oswald et al., 2016).

Each pupil was asked to solve a two-step word problem matched to the age of the pupil through National Curriculum objectives (see below). Each of these problems were designed to lend themselves to being solved using the bar model. The pupils were asked about their self-perception of mathematical ability - another condition deemed significant in mathematical problem solving ability (McLeod, 1985; Morin et al., 2017; Schoenfeld, 1985), along with their overall views of mathematics and problem solving. Finally, each pupil was asked to explain to the researcher how they solved the word problem and why they chose their method.

As neither pupil initially selected the bar model, as a visual representation to support the problem-solving process, both were subsequently asked if they could solve the same problem using the bar model to support them – a process built into subsequent data collection. The pupil discussions were voice recorded and transcribed verbatim (see appendix x). Like the teacher interview schedule, the discussion prompt was guided by the literature in terms of key conditions likely to impact on problem solving performance. The outcome measure for the pupil task was arriving at the correct solution to the word problem.

3.5.1 Calibration of conditions

As QCA is based upon set theory (Atkinson et al., 2020; Schneider & Wagemann, 2010), when defining the conditions for analysis, and the measures associated with them, it is essential that the measure relates to the membership of the set, rather than the opposite condition. For example:



Where the set (a) is specifically defined as those pupils '*not* using the bar model', rather than 'using a strategy other than the bar model' (this example represents a crisp-set).

The iterative process of using theoretical and empirical data relating to the concepts, to determine the level of set membership each condition holds, is known as calibration - a process far beyond that of simple ranking (Jopke & Gerrits, 2019). Along with publication of the raw data from the study, the clarity of the calibration and recalibration process used within QCA is essential to ensure transparency and replicability of the study, and thus constitutes a real strength to the approach (Rihoux & Ragin, 2009; Schneider & Wagemann, 2010).

The initial calibration step requires the identification of the qualitative anchors required for degrees of set membership (discussed below) (Ragin, 2005; Rihoux & Ragin, 2009; Thiem, 2016). Such qualitative anchors refer to 1 (full membership of the set); 0 (full non-membership of the set); and 0.5 (the crossover point – neither fully in nor fully out of the set), as can be seen in figure 23 below.

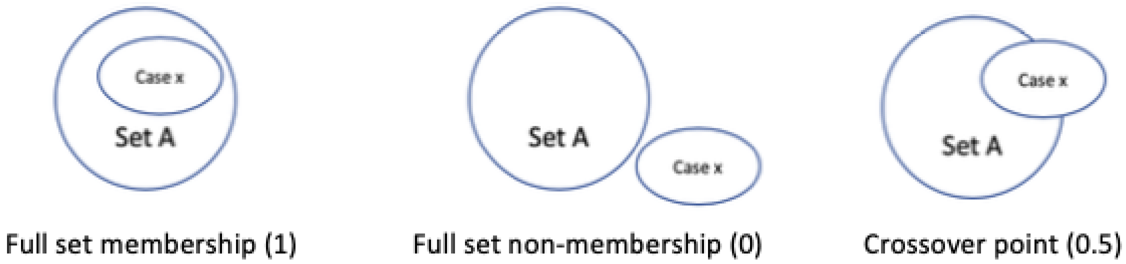


Figure 23: Degrees of set membership for the qualitative anchors

The use of these qualitative anchors (1 – full membership; 0 – complete non-membership; and 0.5 – the crossover point), which is grounded in theoretical and case knowledge, allows for varying degrees of set membership within the conditions. Schneider and Wagemann (2013) argue that the choice of the crossover point, is the most important decision made within the calibration process, in terms of robustness. To maximise the clarity and robustness of the current study, the qualitative anchors used for each condition measure are discussed in detail and grounded firmly on the literature and empirical evidence (see below). Based on these qualitative points and external standards (Hirzalla, 2020; Ragin, 2005; Rihoux & Ragin, 2009), 'Once this process [of calibration] has been completed, the new data are referred to as configurational data' (Thiem, 2016, p. 5).

As QCA is based upon set theory, each condition measure must indicate the degree of set membership for that condition (Eliason & Stryker, 2009; Ragin, 2005; Rihoux & Ragin, 2009; Schneider & Wagemann, 2013). Consequently, the language used for describing the degree of set membership (and its corresponding quantitative measure) must relate to the degree to which the measure resides completely within the set, partially within the set, or not in the set at all. When measuring conditions which are not dichotomous, the number of fuzzy set measures can range from three (1, 0.5, 0) through to a continuous fuzzy set scale, depending on the condition being analysed, and the substantive knowledge gained by the researcher (Rihoux & Ragin, 2009). The use of fuzzy-sets supports a more rigorous analysis, as more detailed information is contained within the sets, strengthening the analysis and setting higher standards for the analysis of subset relations (discussed in section 3.6.2).

Fuzzy Value	**The element is…**
1	Fully in
0.9	Almost fully in
0.8	Mostly in
0.6	More in than out
0.5	Crossover: neither in nor out
0.4	More out than in
0.2	Mostly out
0.1	Almost fully out
0	Fully out

Table 5: Verbal description of fuzzy-set membership scores taken from (Schneider & Wagemann, 2013, p. 29)

When using fuzzy sets, a significant measure, in addition to the degree of set membership, is that of 'negation' – the degree to which the condition is not within the set (Rihoux & Ragin, 2009). For example, if we consider the following theoretical data:

Condition	High self-perception of problem-solving ability (A)	Consistent and correct application of the bar model (C)
Case X	0.7	0.2

Table 6: An exemplar truth table, showing the set membership scores of two conditions for a particular case (case x)

We can interpret that the membership score of Case X to each of these sets is 0.7 and 0.2 respectively. However, when considering negation, the membership *not* in these sets, is 0.3 and 0.8 respectively. That is, the negation is calculated simply as:

a (representing *not A*) = 1 – A

c (representing *not C*) = 1 – C

In addition to negation, based on the concept of Boolean logic, two other key operations are essential to the analysis of fuzzy set data: logical AND, and logical OR – both of which refer to the set relations of the conditions under analysis.

Using the data above, logical AND, and logical OR can be interpreted thus:

If we are interested in those cases which demonstrate a high self-perception of problem-solving ability, and those which consistently and correctly apply the bar model, we are in fact seeking the intersection (combination) of these two sets. As such, the minimum membership score of the sets is used to establish the degree of membership of the case within both sets (0.2).

However, if we are interested in the cases which show either high self-perception, or consistent application of the bar model (known as 'union'), then the maximum set membership score of the component sets is used (0.7) (Rihoux & Ragin, 2009).

Initial condition measures were calibrated primarily using the literature and previous empirical research (discussed in Chapter 3.4.2), before being recalibrated using the data collected from the preliminary study (see Chapter 3.5.2 below). The initial calibration of measures (prior to recalibration), which were based on the current literature, are discussed below and can be seen in Table 7.

Choose to use the bar model

Since one of the main aims of the current research is to find out whether autistic pupils choose to use the bar model, as a visual representation, to attempt to solve a two-step word problem, the first condition for analysis is based upon this choice. As the outcome of this condition has a simple dichotomous outcome (choose to use the bar model or not, irrespective of whether they apply it correctly) the condition is based on a crisp-set measure (discussed above). Consequently, if the pupil chooses to utilise the bar model, then a crisp-set value of 1 is given, and if the pupil does not choose to use the bar model, then a crisp-set value of 0 is given (note again that crisp-set values can be used within fsQCA as discussed above).

Reading age:Chronological age (RA:CA)

The influence of reading ability (both text fluency and reading comprehension) is deemed significant in the overall performance of mathematical problem solving (Björn et al., 2016; Boonen et al., 2013; Kleinert et al., 2015; Oswald et al., 2016). Therefore, one of the measures gained from the semi-structured interview with the class teachers is that of the pupil's current reading age. This measure is considered against the pupil's chronological age to produce a comparative measure, as a quotient (reading age (RA):chronological age (CA)). As reading ability is considered one of the significant conditions in pupils' mathematical problem solving (discussed in the literature review), the use of reading age and chronological age, to provide a quotient, enables a measure to be established with regards to the comparison between the two, clarifying the reading ability of the pupils. As this condition is not a simple dichotomous measure, a fuzzy-set scale is used to establish the initial set membership of this measure. Due to the likely significance of reading ability on pupils' mathematical problem-solving performance, the researcher chose to use the measure of reading age as an indicator of both text fluency and comprehension. Such a measure was made within the pupils' own schools and is based upon a standardised test selected by individual schools, therefore deemed a valid measure of reading age.
Consequently, the initial measures used for this condition were as follows:
Pupil's reading age is greater than their chronological age (i.e. RA>CA) was attributed to a value of 1; pupils' reading age is not greater than, but is the same as their chronological age (i.e. RA=CA) (the cross-over point – neither fully in nor fully out of the set, or that of 'maximum ambiguity' (Ragin, 2005; Rihoux & Ragin, 2009; Roig-Tierno et al., 2017, p. 17)) was attributed a value of 0.5; pupils' reading age is less than their chronological age (i.e RA<CA) was attributed a value of 0.

Current level of mathematical attainment

Again, this concept is far from being able to be measured using a simple dichotomous scale, therefore a fuzzy-set scale was used to measure this condition. Following the abolishment of standardised levels of attainment within primary and secondary schools in September 2014, the current Conservative government acknowledged the need for more freedom for schools in assessing pupils' progress and attainment. The result of this was assessment without levels (McIntosh, 2015), meaning that individual schools were free to measure and record pupils' progress and attainment in their own way. Whilst this provides flexibility to schools, it does remove any consistently applied language and measure to pupils' attainment across different institutions. The initial measures for this condition were based on the language used in DfE guidance from test results in 2018 (DfE, 2018) in order to enhance the external validity of this measure – working at expected level and working at greater depth. Consequently, the 5-scale fuzzy-set measure used to indicate set membership within the set of 'working at greater depth', which relates to mathematical attainment, are as indicated in table 7. The term 'greater depth' was selected here to provide a clear set membership criterion. The cross-over point (0.5) was chosen to be that of the pupil working at the expected level for their current age (hence not fully in nor fully out of the working at greater depth set).

Correct application of the bar model

The measure for this condition was initially based upon the visual representation observation form (VROF) developed in a previous study of pupils' use of visual representations (Bae, 2013). The VROF used in the study by Bae (2013) coded any type of visual representations used by autistic pupils when solving word problems, based on correct spatial relations and solution strategies used within the problem-solving process. In the current study, because reaching the correct solution is a separate measure (the outcome measure), the VROF has been adapted to capture the correct, or incorrect, representation of magnitude and relationships between the known and unknown variables within the word problem, regardless of whether the correct solution was reached. Consequently, this condition is based upon a dichotomous outcome and is therefore a crisp-set measure. Hence, full membership of the set (a score of 1) relates to the pupil using the bar model correctly, representing magnitude and relationships between known and unknown variables accurately. In contrast, non-membership of the set (a score of 0) relates to the pupil using the bar model, however, incorrect representation of magnitude and relationships between the known and unknown variables.

Pupils' self-perception of mathematical ability

The qualitative concept of self-perception of ability is a troublesome condition to quantify due to its subjectivity. The measures used to quantify this concept are based on the Perception of Ability Scale for Students (PASS) (Boersma & Chapman, 1992). The original measure consists of a range of categories in which pupils choose the most appropriate response based on their own self-perception of ability. The assessment has been standardised by reading level for administration from grades 3 (Year 2) to grade 6 (Year 7) (Boersma & Chapman, 1992, p. 1). Furthermore, the assessment has been designed, and used successfully, within a variety of schools and settings (ibid. p.1), hence adding to its validity and reliability. Whilst the original PASS included areas such as reading, mathematics and many others, the current study has adapted the phrases within the mathematics section of PASS to fit the current study. Within the original PASS statements, two specific statements have been used to support the justification of the measure of self-perception of mathematical ability within the current study:

> Statement 66: I am good at maths;
> Statement 51: I am unhappy with how I do maths (Boersma & Chapman, 1992, p. 3).

Subsequently, the measures for self-perception of mathematical ability within the current study utilise a fuzzy-set scale, based around the three qualitative anchors 0, 0.5 and 1 (Ragin, 2005; Rihoux & Ragin, 2009). Full membership of this set corresponds to a high level of self-perceived mathematical ability (I am good at maths) (see table 7, below).

Correct solution reached

The outcome measure for the current study is that of reaching the correct solution to the two-step word problem. QCA analysis of the configurational data from the study will thus enable configurational pathways of conditions, which give rise to the correct solution, to be identified. As with some of the other conditions discussed above, the correct outcome gives rise to a dichotomous measure, hence measured through crisp-set values. Full membership of this set (1) gives rise to the correct solution being reached, whereas full non-membership of this set (0) gives rise to the correct solution not being reached. There is no cross-over point measure within this scale, as the outcome either gives rise to full membership or full non-membership of the set.

	0	0.3	0.5	0.7	1	Notes
Choose to use bar model	*[No]* *Not choose to use bar model*				*[Yes]* *Choose to use bar model*	*This is the first measure to be made. If pupils choose bar model, then application is scored against VROF. If not use bar model initially, then pupils asked if they can solve using bar model and scored against VROF.* ***Not included in fsQCA analysis.***
RA:CA	*<1:1* *Reading age is not greater than chronological age*		*1:1* *Reading age is not greater than, but the same as chronological age*		*>1:1* *Reading age is greater than chronological age*	
Current level of mathematical attainment	*Not working at greater depth, but working below expected level*	*Not working at greater depth, but working towards expected level*	*Not working at greater depth, but working at expected level*	*Almost working at greater depth, but working above expected level*	*Working at greater depth*	*Based on DfE report on Y6 SATs outcomes in 2018 (DfE, 2018b)*
VROF	***Does not use the bar model correctly*** *- Uses the bar model, however, incorrect representation of magnitude and relationships between known and unknown variables.*				***Uses the bar model correctly****, representing magnitude and relationships between known and unknown variables accurately.*	*Validated and efficacy tested through previous study (Bae, 2013b) and adapted for the current study.*
Pupil's self-perception	*Not a high level of self-perceived mathematical ability (I am unhappy with how I do maths (PASS scale no. 51 (Boersma & Chapman, 1992, p. 3)))*		*Not a high level of self-perceived mathematical ability, but not particularly unhappy with mathematical ability*		*High level of self-perceived mathematical ability (I am good at maths (PASS scale no. 66 (Boersma & Chapman, 1992, p. 3)))*	***Draw on PASS study (Boersma & Chapman, 1992)***
Correct Solution reached	*Correct solution not reached*				*Correct solution reached*	***Outcome measure***

Table 7: The initial calibration of condition measures used within the preliminary study.

Using the measures outlined within table 7 above, the data collected for each of the participants within the preliminary study were aggregated to form the configurational data in table 8 below (Thiem, 2016).

	Yr Group	**Choose to use bar model (on initial attempt)**		**RA:CA**	**Current level of mathematical attainment**	**VROF**	**Pupil's self-perception**	**Correct Solution**
P1	*6*	*0*		*1*	*0.7*	*1*	*0.5*	*1*
P2	*3*	*0*		*1*	*0*	*0*	*0*	*0*

Table 8: Data collected from the preliminary study against each of the measures

According to the interviews with both class teachers, and in line with the selection criteria for schools used within the study discussed earlier (where the bar model has been used for at least one year), both pupils had been exposed to the bar model for most of their schooling (P1 for 5 years and P2 for 3 years). The bar model is used within the school as one approach to mathematical problem solving. Although frequently referred to within mathematics lessons, pupils are also taught, and encouraged, to use alternative approaches. Both teachers stated that their aim is to expose pupils to a range of approaches (including the bar model) and to support pupils in selecting the most appropriate approach based on the calculation or problem.

Initial analysis focused on the pupils' choice to use the bar model as an approach to solve the problem. The problems used were as follows:

Year 3: Workmen laid 106 m of pavement a day from Monday to Thursday and 100m on Friday. How many metres did they lay in a week? *(Links to National Curriculum objectives: Y3: solve problems, including missing number problems, involving multiplication and division, including positive integer scaling problems and correspondence problems in which n objects are connected to m objects).*

Year 6: According to one estimate, the Isle of Man has a population three times that of Gibraltar. The two territories have a total population of 120,000. Find the population difference between the two territories. *(Links to National Curriculum objectives: Y6: Solve addition and subtraction multi-step problems in contexts, deciding which operations and methods to use and why; Solve problems involving addition, subtraction, multiplication and division).*

In both cases (n=2), pupils did not choose to use the bar model on their initial attempt at the problem. When using alternative methods (mental calculation (P1) and column addition (P2)), both pupils reached the incorrect solution to the respective problems. Interestingly, pupil P1 was able to recognise that he had not reached the correct solution: 'So…erm, I thought…that…oh wait a sec! Oh god! Why do I…I did that wrong.' (see transcript line 182 in Appendix x). When asked if the bar model may be an approach to help find the correct solution, his response of *'Bar model…yeah! I probably could! It's a picture way of seeing it…and, it's more clear! So, if it's more clear…then you can see it in front of you…which makes sense*' (see Appendix x), demonstrated his understanding of the potential benefits of using visual representations to support problem solving. When using the bar model, pupil P1 immediately represented the magnitude and relationships of the variables provided in the word problem accurately to reach the correct solution (see Appendix xi).

In contrast, pupil P2 felt certain that the correct solution had been reached both when using column addition initially, and the bar model following prompting. When using the bar model, the initial error in retrieving all the information from the word problem on his initial attempt was carried forward, resulting in incorrect representation of size and magnitude of variables within the bar model (see Appendix xii).

When analysing the transcript of the discussion with pupil P2, it appears that there was some lack of verbal understanding in parts of the discussion. For example, when asked about solving problems in maths, pupil P2 began comparing to an alternative context: *'like someone has lost their gloves and can't find them anywhere […] it's like a hard problem').* Furthermore, it was evident from the discussion, that pupil P2's attention was easily diverted to irrelevant information, for example the pencil. This difficulty in focusing on specific tasks and information is consistent with some of the cognitive difficulties typically faced by autistic individuals, as discussed above in the literature review.

Perhaps most notable, was the error made by pupil P2 – simply focusing on the two number values given within the question (see appendix xii) and potentially coupled with the reluctance to use the bar model as a visual representation to support the problem-solving process, rather than the global information provided. This is consistent with the theory of weak central coherence discussed earlier. Pupil P2 exhibited difficulties in processing the global context of the question (a week), to correctly reach the solution. This continued to be evident even when the solution was modelled to the pupil.

When attempting to utilise the bar model to solve the problem, discussion with pupil P2 suggests he had a procedural understanding of the bar model, rather than a conceptual understanding – *'I think I write 106 on top?'* and *'what goes at the bottom squidgy line?'* Such comments made by pupil P2 during his construction of the bar model (albeit incorrect representation of the context of the problem), may suggest that he is familiar with the processes involved in applying the bar model – drawing some bars and drawing 'squidgy lines', but not confident with the reasons for these representations.

Similarly, when applying the column method, it was evident that pupil P1 knew how to 'set out' the calculation, however, he did not show an awareness of working from right to left and the concept of exchanging ten ones for one ten within the calculation. Such examples of instrumental learning align with Siegel's (2009) social cluster deficits discussed earlier (see section 2.2.3).

The findings from this preliminary study provides some evidence to suggest that the bar model provides a useful resource in supporting mathematical problem solving for autistic pupils. Furthermore, the preliminary data aligns with some of the literature discussed earlier (see literature review), which suggests that reading and mathematical ability may be significant in autistic pupils' overall mathematical problem solving ability (Bae, 2013).

When analysing the errors from pupil P2, the difficulty in extracting all relevant information required from the problem, to reach the correct solution, may be linked to difficulties with EF or WCC. The compensation strategies used to maintain behaviour and attention could impact upon the EF skills required to plan and process the solution path required to solve the word problem (Livingston et al., 2018). Similarly, as pupil P2 focused specifically on the numerical values within the word problem, failing to consider the global information providing the context (the amount of pavement laid in a week – see Appendix xii), this may be indicative of difficulties with WCC becoming apparent and impacting on his overall problem-solving ability.

Consequently, the preliminary study enabled the conditions for analysis to be used as part of the recalibration process (see chapter 3.5.3) as well as the trialling and development (discussed below) of the research instruments and design for the main study.

3.5.2 Refinement of research design

In terms of the recalibration of conditions (discussed in Chapter 3.5.3 below), one measure presented difficulty in obtaining the relevant data to match the originally planned measures – reading age (RA:CA). In both cases, data for the current reading age of the pupils was unavailable as the school only assesses pupils' reading age at key points throughout the child's education (hence the data was based on standardised tests completed significantly prior to data collection). Furthermore, it was deemed that this measure did not accurately represent the pupils' reading comprehension ability, which according to the literature (Bae et al., 2015; Björn et al., 2016; Boonen et al., 2013; Jones et al., 2009; Oswald et al., 2016; Özsoy, 2015; Wei et al., 2015; Whitby & Mancil, 2009), is a more accurate predictor of mathematical problem-solving ability. Consequently, an up-to-date reading age was not available for either pupil. During collection of the background data for each pupil (obtained from the respective class teachers), the only current data available with respect to reading was their level of attainment – working at greater depth for both pupils. Consequently, based on this preliminary data, recalibration of this condition (see section 3.5.3) was made to match that of the measures used for mathematical attainment, as this data was available from the school in its current form. Based on this recalibration, the scores for both pupils in the preliminary study remained at 1 as they are both working at greater depth within reading.

Whilst a rich data set was collected from the preliminary study, initial data analysis identified some key aspects for further in-depth analysis, through modification of the research instruments. As weak central coherence appeared to be a potential factor impacting upon the performance of P2 (see above), further data collection, to provide a richer data set for each of the cases, was established through the adaptation and extension to the current research instruments, as discussed below. Furthermore, the additional case-based data provided an opportunity to triangulate the findings with the QCA analysis.

Teacher interviews

To establish a more comprehensive data set on the background of each pupil (case), an additional set of questions, relating to skills encompassed within the executive functions and indicators of weak central coherence, were added.

As discussed within the literature review, the executive functions are accountable for processes such as attention switching, lack of inhibitory control, the need for sameness, decision making, working memory, set-shifting, organisation and planning skills, and generation of ideas (Goldstein & Naglieri, 2014; Hill & Frith, 2003; Lecce et al., 2019; Levy, 2007; Ozonoff et al., 1991; Roelofs et al., 2015; Wen, 2018; Ziermans et al., 2017). Consequently, additional questions relating to individual pupils' classroom behavioural characteristics, which may be linked to EF skills, were added to the teacher semi-structured interview schedule (see below). As discussed in the literature review, in line with the theory that although a set of inter-related skills, the EFs tend to fractionate, and can therefore be distinguished from one another as children progress beyond early childhood (Keenan et al., 2019; Lecce et al., 2019) - the revised teacher interviews were designed to capture such fractionation of skills. The following questions, which were derived from the literature on EF, were added to the interview schedule. The specific aspect of EF potentially linked to each question is also given:

- *What is X's concentration and attention to tasks like generally? Is he easily distracted (if so, by what?), or does he remain on task? Does he often seem to daydream? (EF – attention, set shifting and inhibitory control)*

- *Does X often call out in lessons? (EF – inhibitory control)*

- *Does X appear to struggle with lots of information? Retaining information? (EF – working memory*

- *What are X's organisational skills like? Able to follow instructions? Generally organised with equipment/kits, etc? (EF – organisation (and planning) skills)*

- *What is X's ability to generate ideas/suggested answers to questions like in class? (EF – word and idea generation/planning and decision making)*

To gauge more effectively the use of the bar model within individual teachers' practice, a further question, which probes for more detailed information on this aspect, was also included in the new schedule:

- *Details of use of bar model within mathematics lessons e.g. scheme, how embedded? Frequency?*

The preliminary study enabled the research instruments to be validated for efficacy, along with enabling the identification of key aspects considered worthy of further investigation. Consequently, the research instruments and design were reviewed and developed in line with these findings. The iterative process of gaining a rich insight into the cases within QCA, enabled the researcher to review the data collected and return to the field to gather further data, based on the initial leads.

The current findings were used as part of the main study to see if the initial trends and findings continued to be the case within a larger sample size and across a range of settings.

Most importantly, from this small-scale, preliminary study, the data confirms the use of QCA as a research design, strengthened by its case-based approach, as appropriate tool for data analysis, which will hopefully yield encouraging findings.

Based on the additions and amendments discussed above, the cases from the preliminary study (N=2) were revisited by the researcher to gain this additional information, and data for all subsequent cases was gathered using the amended versions of the research tools.

3.5.3 Recalibration of conditions

Based on the findings from the preliminary study, the only condition measure requiring recalibration was that of reading attainment. As discussed above, the quotient of pupils' reading age against their chronological age was deemed both difficult to access reliable, up to date, data and did not accurately represent the pupils' level of reading comprehension – considered to be a key condition in mathematical problem solving. As discussed, the recalibration of this condition was based on the same measures used to measure pupils' mathematical ability – using the language commonly utilised within schools relating to working at age-related expectations or at greater depth. All other calibrated condition measures used within the preliminary study were appropriate. Therefore, the final recalibrated condition measures used within the main study are presented in table 9 below.

	0	0.3	0.5	0.7	1
Choose to use bar model	*[No]* *Not choose to use bar model*				*[Yes]* *Choose to use bar model*
Current level of reading attainment	*Not working at greater depth, but working below expected level*		*Not working at greater depth, but working at expected level*		*Working at greater depth*
Current level of mathematical attainment	*Not working at greater depth, but working below expected level*	*Not working at greater depth, but working towards expected level*	*Not working at greater depth, but working at expected level*	*Almost working at greater depth, but working above expected level*	*Working at greater depth*
Visual Representation Observation Form (VROF) adapted from Bae (2013)	***Does not use the bar model correctly*** *- Uses the bar model, however, incorrect representation of magnitude and relationships between known and unknown variables.*				***Uses the bar model correctly****, representing magnitude and relationships between known and unknown variables accurately.*
Pupil's self-perception	*Not a high level of self-perceived mathematical ability (I am unhappy with how I do maths (PASS scale no. 51 (Boersma & Chapman, 1992, p. 3)))*		*Not a high level of self-perceived mathematical ability, but not particularly unhappy with mathematical ability*		*High level of self-perceived mathematical ability (I am good at maths (PASS scale no. 66 (Boersma & Chapman, 1992, p. 3)))*

Table 9: The final recalibrated measures used for conditions for analysis within QCA (note the inclusion of some crisp-set values for some conditions

One of the main processes for strengthening the trustworthiness of QCA, is the necessity to ensure rigour and transparency throughout all stages of the research design. The use of clear calibration (and recalibration) as in the current study is one method to ensure this. This enables all decisions to be clear and justified by theoretical and case-based evidence, and that all stages of the process are justified and fully explained (Rihoux & Ragin, 2009; Schneider & Wagemann, 2013).

3.6 Data Analysis

3.6.1 Data analysis: Coding

According to Schneider and Rohlfing (2016), QCA 'works best in combination with the intimate knowledge of cases' (p.557) and should ideally be used alongside intensive case study, rather than instead of (Legewie, 2013). Consequently, this section discusses the qualitative analysis of the data from the current study. The generation of codes and subsequent analysis is discussed. The in-depth knowledge of the case data is significant for understanding the underlying mechanisms giving rise to the outcome (correct mathematical problem solving). Furthermore, through such analysis, similarities and negative cases can be identified to support the QCA analysis.

Prior to analysis with fsQCA (Ragin & Davey, 2016) (discussed below in 3.6.2), the data for each case was aggregated and coded using NVivo 12 software (QSR International Ltd, 2018) (see figure 24). The data was coded using both inductive and a priori coding to identify factors relevant to the specific conditions for analysis, along with data indicative of behaviours associated with autism, providing a clear representation of the background data for each case. Such an inductive approach complements QCA methodology, as it seeks to explore data sets, without testing any specific theory, as in the case with the current study (Hirzalla, 2020).

Within the current study, the use of NVivo 12 software (QSR International Ltd, 2018) was used as one approach to support the analysis of the significant volume of qualitative case data collected. The use of computer assisted qualitative analysis software (CAQDAS) supports the development of an holistic, data rich profile for each of the cases within the study and supports visibility and clarity within the coding process, thus enhancing rigour and reliability to the process of data analysis (Tummons, 2014). Through the integration of data analysis from a wide range of sources (pupil interviews, teacher interviews, pupils' completion of the mathematical problem-solving task and the sentence completion task, all discussed within the methods chapter), a comprehensive case profile was developed for each case.

To maintain the closeness between the researcher and the data, coding within NVivo 12 consisted of both *a priori* codes (driven by the literature), as well as inductive codes, which were derived from the analysis of the individual and aggregated data sets. Furthermore, all interviews were transcribed verbatim by the researcher, to support the closeness between the researcher and the data. The use of CAQDAS to analyse the aggregated data for each case, supports the identification of coded segments of both text and pupils' mathematical workings out, across a wide range of individual case data. Through the analysis of codes applied to the data set within each case, the frequency, and therefore, potentially, the significance of specific codes becomes clearer, hence drawing the researcher closer to the case data. Each individual piece of data, once coded to the respective case (pupil) within NVivo 12, was then analysed and coded according to the *a priori* codes generated prior to any data analysis. The *a priori* codes were based on the initial conditions for analysis identified prior to data collection and driven by the literature (see figure 24 below). However, critics of *a priori* coding within social science studies argue that such a strategy may restrict 'accurate observation', and should therefore be avoided (Kettley, 2012, p. 95). The critics' arguments rest on the argument that 'society is composed through human relationships' (ibid. p.95) and hence the use of *a priori* codes may steer the researcher away from observing these relationships. Nevertheless, due to the methodological framework of the current study, the use of *a priori* coding is justified in terms of seeking the underlying mechanisms, which may give rise to the correct solution to a mathematical word-problem, through the analysis of theory-driven conditions for analysis.

Following *a priori* coding, inductive coding was used to code any aspects, which the researcher considered significant to the analysis, that had not been considered *a priori* (see figure 24 below). An example of one such inductive code is that of 'prompting'. During analysis of the interview transcript of case SchGP1, it emerged that the pupil had potentially been prompted by his teaching assistant prior to the interview with regards to the use of the bar model. The codes – both *a priori* and inductive, used and generated from the data analysis within NVivo 12, can be seen in figure 24 below. As the coding process developed, such top-level codes, become the categories within the data analysis, within which codes reside. Through the combination of techniques of *a priori* and inductive coding of the data, it is anticipated that the researcher may begin to identify the mechanisms at play within the domains of the empirical, the actual and the real (Kettley, 2012). The aim of such initial data analysis is to identify any regularities within the data, which support the 'detection of deep causes of social formations' (ibid. p.142).

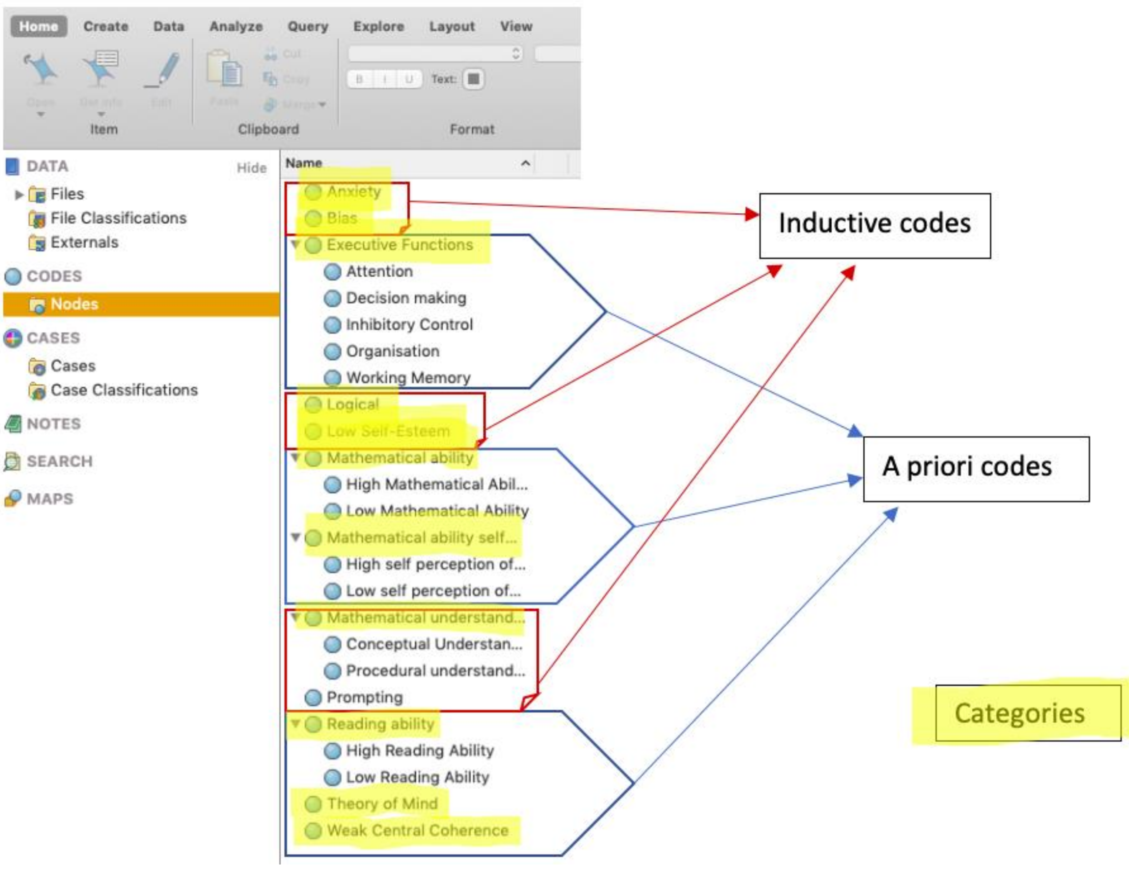


Figure 24: A priori and inductive coding used within NVivo 12 coding

As the current study is based on a critical realist framework, the use of detailed case-based profiles, generated through data analysis within NVivo 12, supports the identification of potential generative mechanisms underlying the configurational pathways identified through fsQCA analysis (discussed in chapter 3.6.2).

Following the detailed coding of all data for each case, a case profile was generated (see figure 25 below) to provide a visual overview of each case. The detailed coding used within NVivo 12 was used to complement the QCA analysis (discussed chapter in 3.6.2), supporting the iterative process of data analysis. The identification of the configurational pathways within fsQCA, coupled with the detailed case data, generated from NVivo 12 analysis, enabled the researcher to delve deeper into each case to analyse and identify trends and supporting explanations for the configurational pathways established. In addition, the use of cross-case analysis can be supported through analysis of the detailed case-based profiles built up for each case.

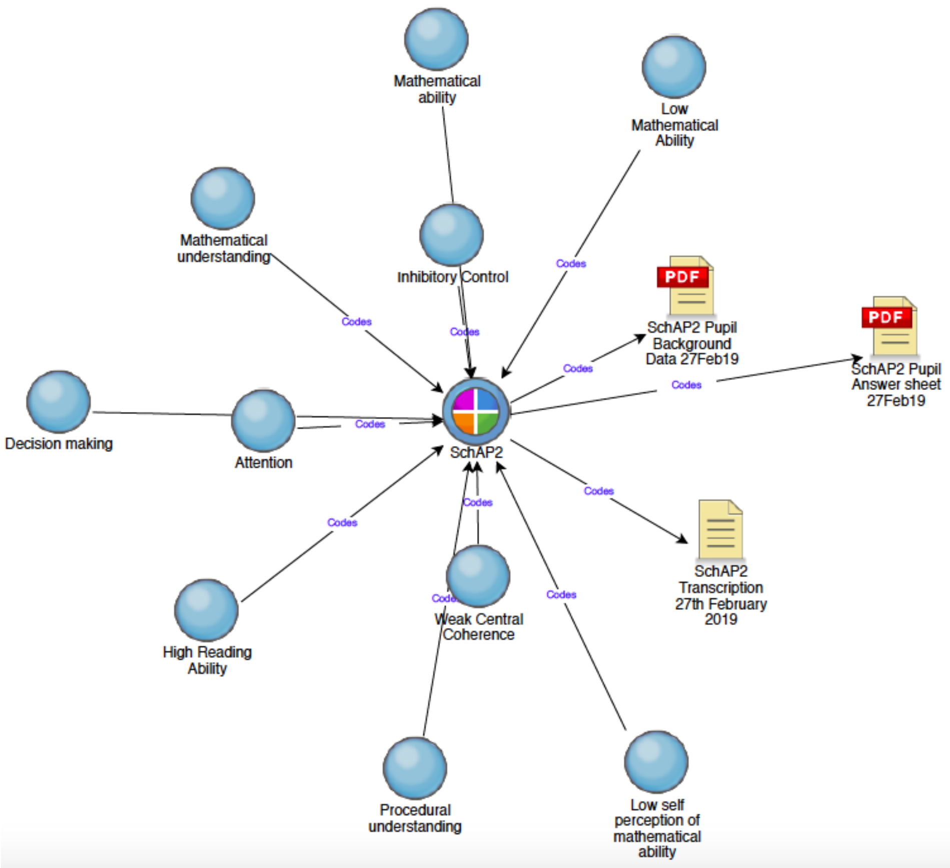


Figure 25: Example of case-based analysis using NVivo 12 (SchAP2)

3.6.2 Data analysis: QCA

The research reported here is complex and so the following section attempts to mitigate for this by guiding the reader through the detailed analysis of the data using QCA. The current section supports the reader in understanding the data analysis process and techniques applied to each of the data sets collected from the individual research instruments employed within the data collection. Throughout, a detailed explanation of the analysis of the data set using QCA is provided, enabling the reader to follow the complex analytical steps carried out.

To support the reader and to provide clarity, a brief explanation of the key terms in QCA used throughout the remainder of this chapter and throughout chapter 4 are provided below:

Complex (or conservative) solution - derived entirely from the truth table rows (i.e. cases), which demonstrate sufficiency for the outcome of interest. It is therefore based wholly on the empirical evidence available from the observed cases and no logical remainders are considered. Consequently, it provides a subset of the other two solution types.
Consistency – the degree to which the empirical data are in line with the suggested subset relation.
Coverage – a measure of how much of the outcome set is covered by the condition set.
Fuzzy-set – a set allowing full, partial, or complete non-membership.
Intermediate solution - a solution, which is entirely based upon simple counterfactuals, where assumptions are made about those combinations of conditions for which there is no empirical data set, but based on the data from the observed cases, a logical assumption can be made as to the likely outcome (based on those logical remainders which are most plausible).
Inus condition - a single condition, which is insufficient for producing the outcome on its own, but which is a necessary part of a conjunction (or combination of conditions) which in itself is unnecessary but sufficient for the outcome (Schneider & Wagemann, 2013, p.328).
Limited diversity – where there is insufficient empirical evidence to support all of the logically possible combinations of conditions) within QCA (Rihoux, 2013, p. 240).

Logical minimisation – summary of information provided within the truth table, through removal of redundant conditions.
Logical remainder – Any truth table row, where insufficient empirical data is available.
Minimum formula – configuration of conditions following the removal of redundant conditions.
Necessary condition – 'always present when the outcome occurs (i.e. the outcome cannot occur in the absence of the condition)' (Rihoux & Ragin, 2009, p. xix).
Parsimonious solution - a solution based on both the observed cases (empirical data) and the logical remainders, however, unlike the intermediate solution, the plausibility of the logical remainders is not considered.
Sufficient condition – 'always occurs when the outcome is present. However, the outcome could also result from other conditions.'(Rihoux & Ragin, 2009, p. xix).
Truth table – sorts cases according to one of the logically possible combinations of conditions.

Schneider & Rohlfing (2016) propose that the cross-case analysis within QCA is most successful when accompanied with a case study approach, similar to that discussed above within the current study. This complementary approach enables further interrogation of the causal pathways to take place, to explore the underlying mechanisms. Furthermore, through the constant dialogue with the cases used within QCA, exploration of any conditions excluded from the analysis, can be carried out to consider their impact on the overall outcome. Consequently, the data analysis using QCA, was used to complement those data analysis approaches discussed above to support an in-depth interpretation of the data from cases within the study, further supporting the rationale for a small-N, which ensured an in-depth case knowledge to be gained.

Data analysis within QCA draws on the concept of identifying 'demi-regularities' within the data at the empirical level (Fletcher, 2017; Lawson, 1997). Here, the concept suggests semi-predictable outcomes, based on the CMO configurations discussed earlier (i.e. acknowledging the unpredictability of the influence of the social world). These demi-regularities are considered as 'tendencies' rather than laws (as discussed earlier), as it is assumed that the influence of the social world on events simply cannot support the deterministic predictions employed within many causal claims (Lawson, 1997, p.185). Consequently, these 'tendencies' imply that a mechanism is actualised sometimes, but by no means in a universal manner (Archer et al., 2007; Lawson, 1997). When applied to the current research, such demi-regularities established within the literature review, were used to provide the rationale for conditions for analysis within the present study (discussed in chapter 3.4.2). Fletcher (2017) refers to this as a more 'theory- and researcher-driven analytical process' (p.186), rather than the data-driven approaches used, for example, in grounded theory analysis.

In addition to the identification of 'demi-regularities', the process of 'abduction' (theoretical redescription), enables the data to be 'redescribed using theoretical concepts (p.188) whilst still acknowledging the fallibility of the theories' (Fletcher, 2017, p. 188). From a critical realist approach, a mode of inference known as 'retroduction' (Danermark et al., 2002; Fletcher, 2017) is considered, where our knowledge reality extends beyond observable events (the empirical data) to question the deeper levels of reality in which the generative mechanisms are operating (within the domains of the actual or the real). The aim of this step within the data analysis process of the current study is to identify the 'necessary contextual conditions' (Fletcher, 2017, p. 189) involved within the mathematical problem solving process for autistic pupils. Maxwell (2012) supports this approach by discussing the importance of data being interpreted within its contextual background (Maxwell, 2012), thus warns against categorising and coding strategies, which may 'decontextualise' the data, but instead considers the use of 'connecting strategies' within data analysis. This is where 'categories, or 'phenomena', are specified in terms of the conditions that give rise to it [and/or] the context in which it is embedded' (Strauss & Corbin, 1990, p. 97). Here, the links between the coding strategies discussed in Chapter 3.6.1 above, and the current analysis using QCA, become clear within the present study.

At this point, it is worth clarifying the concept of 'necessity', in line with the philosophical framework underpinning the current research, which is provided by Archer et al. (2007):

'To attribute necessity to […] a condition, an outcome or effect, […] we contend to indicate that within the relevant context no alternative to that condition, outcome truth-value or conclusion is possible' (pp.113-114).

Consequently, any conditions considered as necessary within the current study, are therefore deemed to be in alignment with the above conceptualisation of the term.

However, as Thiem (2018, p.3) points out, 'just because some factor does not appear in the final model, does not mean that this is causally irrelevant to the analysed outcome; it simply means that, conditional on the available data, there exists no evidence for the causal relevance of that factor.' Therefore, within the current study, it is important to acknowledge this argument, as is discussed further in Chapter 6.

Following data collection, the conditions for analysis, discussed in the Chapter 3.4.2, were coded according to the predetermined, calibrated and recalibrated measures discussed earlier (see table 9 in chapter 3.5.1).

- Pupil's current level of reading attainment;
- Pupil's current level of mathematical attainment;
- Choose to use the bar model to solve mathematical word problems;
- Visual representation application and accuracy;
- Pupil's self-perception of mathematical problem-solving ability.

Figure 26: The conditions for QCA analysis within the current study

All measures were based on a fuzzy-set scale (discussed earlier), resulting in a measure ranging from zero (complete non-membership of the set) to one (completely membership of the set). Again, those conditions with binary outcomes (i.e. crisp-sets) are (and can be) still used within fuzzy-set analysis.

QCA analysis was carried out using fsQCA version 3.0 software (Ragin & Davey, 2016). Unlike the analysis of case data discussed above, analysis within fsQCA 3.0 (Ragin & Davey, 2016) was carried out using the two subgroups within the study: the data pertaining to those cases where the pupil was autistic (N=7) and the data pertaining to those neurotypical pupils (N=2). As the current study is concerned with the identification of any necessary or sufficient conditions associated with mathematical problem solving for autistic pupils, only the analysis from the autistic subgroup (n=7) is discussed in detail. Here, analysis of the data pertaining to the neurotypical group only, results in significant limited diversity (discussed earlier), where the number of possible configurations, in this instance significantly, outweighs the amount of empirical data available. Furthermore, analysis of both subgroups as one data set (N=9), would provide little (if none) indication of the significant conditions pertaining to mathematical problem solving for the autistic subgroup, of which the focus of the current study is concerned. In contrast, the detailed analysis of the case data discussed earlier, was key to the current study, in terms of exploring the potential mechanisms (such as the influence of the EFs and WCC) giving rise the observed outcomes, through comparative analysis between the two subgroups. Within fsQCA 3.0 analysis (Ragin & Davey, 2016), the outcome measure was always the correct solution, as the study seeks to identify the configurational pathways and any sufficient (or necessary) conditions required, including the significance of the bar model as a visual representation, for reaching the correct solution to the mathematical word problems.

Using fsQCA 3.0 (Ragin & Davey, 2016), three separate analyses were carried out on the data sets: analysis of necessary conditions; set coincidence analysis; and subset/superset analysis. Analysis of necessary conditions enables the researcher to identify both individual conditions, and the multiple configurational pathways of conditions, which suggest a necessary condition(s) for the solution (in this case, reaching the correct solution to the mathematical word problem). Set coincidence enables the researcher to obtain a measure of the degree of overlap of two or more sets pertaining to the solution outcome. Finally, subset/superset analysis enables the identification of those conditions (sets) which are either part of (subset) or contain other sets (supersets) pertaining to the outcome measure. For all analyses, a default consistency score of 0.8 was used as a cut-off, as recommended in the fsQCA 3.0 user's manual (Ragin, 2017, p. 40). Such consistency scores relate to 'the percentage of a cases' set-membership scores in two sets that is in line with the statement that one of the two sets is a subset (or superset) of the other. It thus indicates the degree to what the empirical data are in line with a postulated subset relation.' (Schneider & Wagemann, 2010, p. 324). As discussed earlier, the use of the 0.8 cut-off for consistency, adds to the rigour of QCA analysis (Schneider & Wagemann, 2013).

Analysis with fsQCA 3.0 (Ragin & Davey, 2016) generates three solutions types – the intermediate, the parsimonious and the complex (or conservative) solutions. The intermediate solution provides a solution, which is entirely based upon simple counterfactuals, where assumptions are made about those combinations of conditions for which there is no empirical data set (i.e. logical remainders, as discussed in Chapter 3.2), but based on the data from the observed cases, a logical assumption can be made as to the likely outcome (based on those logical remainders which are most plausible). In contrast, the conservative solution, is derived entirely from the truth table rows (i.e. cases) (discussed below), which demonstrate sufficiency for the outcome of interest. It is therefore based wholly on the empirical evidence available from the observed cases and no logical remainders are considered. Consequently, it provides a subset of the other two solution types. Finally, the parsimonious solution, provides a solution based on both the observed cases (empirical data) and the logical remainders, however, unlike the intermediate solution, the plausibility of the logical remainders is not considered.

According to Thiem (2016), appropriate selection of solution type is crucial – he argues that the parsimonious solution is the most appropriate, despite some methodologists within the field stating otherwise (Ragin et al., 2008; Schneider & Wagemann, 2013). The reason for this being that in line with inus theory, discussed earlier, unlike the parsimonious solution, the conservative and intermediate solutions artificially inflate the data and can produce models that claim causal relevancies for which there is no evidential basis (Baumgartner & Thiem, 2017; Thiem, 2016) (see discussion, chapter 5). However, Rihoux and Ragin (2009), conversely argue that the intermediate solution is 'superior' to the other two, as it does 'not allow for the removal of any necessary conditions' (p.111). Therefore, within the current study, the intermediate solution is selected as the preferred choice, in line with Rihoux and Ragin's (2009) arguments. However, regardless of the solution type used, following minimisation, it is essential that any such solution is interpreted within the context of the cases within the study (Hirzalla, 2020), hence the complementary analysis approaches within the current study.

In order to enhance the rigour and transparency of the data analysis process, the inclusion of all data, in representational form, is advised (Schneider & Wagemann, 2013). Based on this advice, the current study provides the reader with the complete data sets throughout the discussion of data analysis and findings. Furthermore, despite using the intermediate solution (discussed above) within the current study, all solution types are presented to the reader to provide clarity over the data analysis. In line with Schneider and Wagemann's (2013) suggestion, similarities in consistency and coverage between the different solution types indicate robustness of the data (discussed further in Chapter 4.6).

The first step in the analysis of configurational pathways, was to construct a table of the raw data in terms of fuzzy sets, using the calibrated (and recalibrated) condition measures discussed in Chapter 3.5.1. Case codes were also omitted at this stage, as they are not required for QCA analysis. Note also that the measure for VROF (visual representation observation form) is omitted from the analysis with fsQCA 3.0 software. The rationale for this is that in analysis prior to the use of fsQCA 3.0 (Ragin & Davey, 2016), the need to correctly represent magnitudes and relationships of the numbers from the word problem, when using the bar model to reach the correct solution, was identified (see Chapter 4.4 below). Furthermore, the reduction in the number of conditions for analysis, in relation to the number of cases analysed, reduces the likelihood of limited diversity, as discussed earlier.

Using the calibration of conditions discussed in the chapter 3.5.1, the raw case data was then converted to fuzzy-set values. Table 10 below, reminds the reader of the threshold membership values and cross-over points used within the calibration process. Note the change in notation when referring to the source variable (e.g. RAtt) and when referring to the fuzzy-set (RAttfz).

Fuzzy Set	Source Variable	Threshold for full membership	Cross-over point	Threshold for full non-membership
RAttfz	RAtt	Greater Depth	ARE	Significantly <ARE
MAttfz	MAtt	Greater Depth	ARE	Significantly <ARE
BM+fz	BM+	Yes	N/A	No
SPerfz	SPer	High	Average	Low
CorrSolnfz	CorrSoln	Yes	N/A	No

Key to table x
RAtt: Reading attainment
Matt: Mathematical attainment
BM+: Use of the bar model
SPer: Self-perception of mathematical ability
CorrSoln: Correct solution reached
(xxx)fz: fuzzy-set condition
ARE: Age-related expectation

Table 10: Calibration of the data into fuzzy-sets

In order to identify the subset relations of each source variable (condition) to the outcome (reaching the correct solution), an XY plot, for each source variable was made within fsQCA 3.0 software (Ragin & Davey, 2016) (see Chapter 4.6). The XY plot generates a consistency score, indicating the degree to which the data supports a subset relation of each condition to the outcome measure. As it is quite possible for the absence of a condition to form part of a configurational pathway for the outcome, negated values (absence of the condition) were also plotted against the outcome measure. Following the identification of the consistent subset relations (those with consistency scores >0.8), subset/superset analysis was carried out using fsQCA 3.0 (Ragin & Davey, 2016). Detailed analysis of the data was carried out using QCA, based on the identified conditions, which were subsets of the outcome (correct solution) based on consistency scores of <0.8. Each of the conditions were analysed in terms of their presence, or absence (negated), against the outcome.

Following the creation of the data matrix, fuzzy data matrix and identification of all possible configurations of these identified conditions (tables 23, 24 and 25 in Chapter 4.6), the membership of all cases in all configurations was established (table 26).

For each of the non-remainders, the outcome values are next determined based on the raw consistency score of each truth table row. The consistency for sufficiency of a configuration (comparison of the membership of all cases in the configuration of interest with the membership of all cases in the outcome) is determined through the formula:

$$\text{Consistency}_{\textit{Sufficient conditions } (Xi \leq Yi)} = \frac{\sum_{i=1}^{1} \boldsymbol{min}(\boldsymbol{Xi}, \boldsymbol{Yi})}{\sum_{i=1}^{1} \boldsymbol{Xi}}$$

(X refers to the membership score of the condition, Y refers to the membership score of the outcome) (Schneider & Wagemann, 2013, p.126)

This comparison is relevant as consistency is indicated when membership scores in the configuration is consistently less than, or equal to, membership scores in the outcome. The consistency scores were used to generate the final truth table (see table 29 in Chapter 4.6), in preparation for analysis of the solution formula in fsQCA 3.0 (Ragin & Davey, 2016).

At this stage, logical minimisation (discussed earlier) was carried out to compare the sufficient configurations and to result in a minimum formula for sufficiency. Through this process, redundant conditions were removed from the configurations to leave the minimum formula. Using this data, truth table analysis was carried out to establish any necessary, or sufficient, conditions.

The final step in the data analysis, using fsQCA 3.0 software, was the analysis of necessary conditions. Analysis of necessary conditions for the data from the current study was carried out only on the autistic cases (n=7), for the reasons discussed above.

As part of the contribution to knowledge for the current study is the use of QCA in small-scale educational research, analysis of configurational pathways emerging from the data were also generated manually. This comparative data was then used to support the critique of QCA in small-N, educational research. As there is a gap in such research using QCA, the manual analysis is used to enhance the reliability of the findings from fsQCA 3.0 analysis within the current study. Whilst the reliability of QCA studies is commonly enhanced through the use of two, or more, researchers carrying out the process (Roig-Tierno et al., 2017), the current study does not allow this, hence alternative methods for increasing the reliability of the data have been considered.

For each case, the configurational pathway was derived by the researcher, using the raw data. For example, case SchAP1 chose to use the bar model (BM+fz); had a reading and mathematics attainment above age-related expectation (RAttfz, MAttfz); and had a high self-perception of his own mathematical ability (SPerfz). When completing the mathematical problem-solving task, he reached the correct solution, therefore the configurational pathway for this case can be denoted as:

BM+fz x RAttfz x MAttfz x SPerfz → Correct Solution

The manual generation of the configurational pathway for each case was then aligned to the correct, or incorrect, solution reached to the mathematical word problem. Consequently, the researcher was able to identify the combinations of conditions (configurational pathways), which gave rise to the correct solution to the word problem.

Through comparing the results of the manual configurational pathway analysis, to those configurational pathways derived using fsQCA 3.0 software, the researcher was able to enhance the validity of the findings from QCA analysis. Furthermore, comparison of the two sets of configurational pathways (manual and fsQCA 3.0-derived), was be used to determine the reliability of QCA as a data analysis approach within small-N, educational research. However, it must be acknowledged here, that the manual analysis does not allow for any logical remainders to be considered, hence must be used with caution.

3.7 Ethics

'Ethics goes beyond ethics committee approval and consent documents. It addresses broader issues of respect, inclusion, and empowerment in the everyday context of research' (Cascio et al., 2020, p.1676). The ethical considerations within the current study sought to acknowledge the awareness of the researcher to working with doubly-vulnerable participants – children and those autistic individuals. The following section outlines the ethical considerations considered, along with the processes and steps put in place to ensure the research was ethically sound, providing the reader with evidence to support responsible research with autistic participants.

As the current study sought to further develop our knowledge of autism, the inclusion of the autistic voice is paramount, to ensure the research is both epistemologically and ethically sound (Chown et al., 2017; Leatherland, 2018, p. 127). Using pupil discussions/interviews, the voices of the autistic participants provided an integral part of the data collection and analysis in order for their personal views on mathematics teaching and learning, and their own performance and confidence in problem solving, to be ascertained. Whilst the current study cannot be deemed as being fully participatory, as 'equality of input for participants at every stage of the research process' was not offered, it does however make the shift away from the widely used medical approach to autism research. In such medical studies, 'research is undertaken on autistic people' towards research 'with them', in an attempt to 'improving the day to day lives of autistic people' (Chown et al., 2017, p.3), which is considered one of the 'most important' features of autism research (ibid., p.13). In the current study, the social model of disability (discussed in chapter 2.1.3) is 'at the heart of the project ethos', where there is a search to 'remove the societal barriers' (ibid., p.12), which may prevent autistic pupils accessing and thriving in the mathematics classroom.

Each of the participating schools within the current study was identified based on either recommendation, or schools known to the researcher to be using the bar model within their teaching of mathematics. Access to each of the settings, as a research base, was gained through an initial liaison and meeting with the head teacher and mathematics leader to discuss the anticipated research and the criteria for case and condition selection (see chapter 3.4.1 and 3.4.2 above). Due to the difficulties in sourcing suitable schools and pupils, which met the criteria for the study, and who were willing to participate, the current study was broadened to include pupils in both Year 6 and Year 2. Although the original PICO criteria (see Chapter 3.4.1) excluded pupils in Year 2 and Year 6 due to the potential influence of forthcoming SATs tests, the limited access to suitable schools and cases has resulted in this exclusion criteria being dropped. However, it is important to acknowledge the potential influence of SATs preparation for these pupils when analysing and interpreting the data. Furthermore, in addition to the ethical considerations presented within the approval for ethical consent for the present study, a further discussion with the head teacher as to the potential additional pressure faced by these pupils was also discussed and must again be acknowledged, in terms of causing increased stress, anxiety and potential communication difficulties (Leatherland, 2018). From the initial meetings with the head teachers and the mathematics leaders, the potential pupils were identified in each school to form the cases. The vulnerabilities of the participants within the current study, as both children and autistic individuals, was therefore acknowledged and reflected upon, in terms of the ethical issues, throughout the research process (Leatherland, 2017). Due to the double-vulnerability of the pupils, an additional trusted adult was present during all mathematical problem-solving tasks to ensure familiarity to the pupils and to help reduce any stress or anxieties. In all cases, the additional adult was familiar to the child (either the teacher or teaching assistant) to reduce any anxiety of introducing another unknown adult to these vulnerable pupils. Furthermore, the presence of an additional, trusted adult ensures that research involving children, such as in the current study, 'takes place in partnership with caring, skilled adults who need to provide appropriate support and guidance, to help them formulate their views and participate in a safe and meaningful way' (Powell et al., 2013, p.18). Consequently, all participants within the study had the option to access additional support from the trusted adult at any point, should they wish to do so, ensuring that any additional needs for participation within the research were met.

When collating the basic background data on the cases within the study, those pupils not meeting the selection criteria were de-selected (as discussed in Chapter 3.4.1).

Ethical approval for the study was granted from Durham University on 30th November 2018 (see appendix xx). An amendment to the ethical approval was granted on 8th April 2019 (see appendix xxi), based on the requirement to include neurotypical pupils and to administer the STEM sentence completion task (which were not in the original application).

For all cases, a participant information form (see appendix xxii) was sent to all parents and written parental consent was sought and gained before any data collection began. In addition to the ethical considerations set out within the ethics application protocol, a further discussion with the head teacher and the mathematics leader sought to ensure the pupils themselves would be comfortable with participating in the study, based on their current behaviours and confidence within the school, to ensure no additional anxiety was caused.

In addition to parental consent, the class teachers were presented with the participant information form and taken through the research and its ethical considerations. Although the teachers were active research participants, their role is to provide background information on the cases (pupils) within the study. Prior to any data collection, all pupils were introduced to the researcher and the requirements of their engagement and participation were explained fully, both by the researcher and their class teacher. Due to the vulnerability of the pupils, a discussion with the head teachers ensured that all pupils were able to understand and make informed decisions regarding their participation in the study. To document this, the pupils agreed to participate fully and signed the pupil consent forms, on the understanding of their rights to withdraw, anonymity and other ethical considerations outlined in the participant information sheet.

As outlined in the ethical application, GDPR regulations were adhered to and all data collected was anonymised and kept on a secure, password protected, external device. Following the publication of the current study, all data pertaining to individual cases will be destroyed.

3.8 Positionality: validity, rigour, and trustworthiness

Although 'general laws of robustness' within QCA studies is difficult to attain due to the limited publication of robustness literature concerning QCA, high standards can be ensured through the clarity and detail provided through the data analysis process, as in the current study (Schneider & Wagemann, 2010, p. 285). Throughout the data analysis (discussed above) and the clarity applied to the interpretation of the data, the robustness of the current study is maximised. As robustness aligns with uncertainty, a rigorous process has been embedded within the current study to maximise the validity of the findings. Despite the small-N used within the current study, based on the rationale discussed earlier, throughout the analysis and interpretation of the data, indicators of robustness have been discussed in detail to demonstrate the acknowledgement of this throughout.

Maxwell (1992) presents three types of validity, which align with a realist view: descriptive, interpretive and theoretical validity (p,134), however accepts that there are 'fuzzy boundaries' between these types of validity. Maxwell's (1992) descriptive validity, on which all other types are dependent, is based upon the 'factual accuracy' of the researcher's accounts, which can be backed up with the data to a point, which can usually resolve issues of validity 'beyond reasonable doubt' (Maxwell, 2012). This is demonstrated in the current study through the transparent recording and presentation of all data pertaining to the analysis and the use of clarification questions and respondent validation, as discussed earlier. This position maintains the epistemological understanding that these accounts are potentially fallible and that the feasibility of absolute certainly is not possible. If descriptive validity can be assured, then interpretive validity, which relates to the meanings of the behaviours of the participants, can be considered. It must be remembered that the phenomena of meanings from the participants, must be considered as real and are interpreted from an emic perspective (that of the participant, rather than the researcher or theoretical abstractions) (Maxwell, 2012). Finally, Maxwell's (2012) third type of validity, theoretical validity, considers the 'theoretical constructions the researcher brings to the study' (p.140). Through clear and precise operationalisation of the constructs used within the current study, such as word problems and problem-solving ability, it is anticipated that these threats to theoretical validity have been reduced throughout.

However, within QCA, Schneider and Wagemann (2013) identify some key aspects, which should be considered and adhered to, maximising the robustness and minimising any ambiguity within the research (most of which have been discussed within the relevant sections of the thesis). As discussed above and is demonstrated within the current study, the transparency and availability of the data recording and analysis process is fundamental to ensuring the reliability and validity of the research. Furthermore, the presentation of all data (including the three solution types – discussed above and presented in Chapter 4.6), is of paramount importance. This enables all data analysis steps to be replicated and an analysis of the comparisons of consistency and coverage between the different solution types to be carried out (Schneider & Wagemann, 2013). Furthermore, the use of a consistency threshold of 0.8 (recommended to be a minimum of 0.75 (Rihoux & Ragin, 2009; Schneider & Wagemann, 2013)) is used within the current study to ensure that the findings are robust. Such use of a high threshold reduces the number of rows (cases) used within the final minimisation process (demonstrated in Chapter 4.6 below) to ensure greater reliability of the final solution outcome, in terms of representation of cases.

The calibration (and recalibration) of condition measures (Chapter 3.5.1), particularly the qualitative anchors, must be firmly grounded in the literature and empirical data, as discussed earlier in the current study. Once again, the transparency of the justification for such measures and calibration provides a replicable data set.

As discussed in Chapter 3.4.1 and 3.4.2, decisions surrounding the selection of cases and conditions are imperative to ensuring robustness and validity of any claims made. These were guided by the previous literature and empirical evidence discussed in Chapter 2. Such case selection decisions are purposive, rather than random, to ensure that the conditions of interest are present (or not) in the selected cases within the study (see Chapter 3.4.1). In terms of internal validity, issues such as reactivity (influence of the presence of the researcher); unawareness of previous events, which may influence the present findings; and power relations between the researcher and the participant, must all be considered (Cohen et al., 2018). The current study sought to minimise these through the in-depth discussions with the head teachers, class teachers and mathematics leaders, as discussed above. The triangulation of the data from semi-structured interview with the class teachers, in order to establish background case data on the pupils, offers a further approach within the current study to address this threat – particularly with respect to prior events which may influence the current findings (Denzin & Lincoln, 1994).

Although some threats to both internal and external validity have been discussed within the context of specific research instruments (above), along with a consideration of the generalisation of causal claims made within QCA, it is important to reinforce the acknowledgement of these for the purposes of the current study. From a critical realist perspective, it is essential to consider the subjectivity introduced to the study by the researcher themselves, for example, their beliefs, values and dispositions (Maxwell, 2012), which can lead to misinterpretation of the data. Whilst many researchers will attempt to reduce, or eliminate the subjectivity within the research, critical realists accept subjectivity as part of the 'actual process of understanding' (p.98), which can be enhanced and developed using reflexive practices throughout the research, particularly in reducing (or ideally eliminating) interviewer bias. Researcher relationships with participants, can have profound effects on the data produced, and thus ultimately the outcomes of the study. However, within a realist framework, these relationships are considered as real phenomena which are influenced by the context within which the research is conducted (Maxwell, 2012), therefore directly influencing the outcomes of the research and the research process. At this point, we can consider the CMO configurations discussed earlier within the methodology chapter to fully understand this claim made by Maxwell (2012).

Often referred to as external validity, generalisability, refers to the applicability of the research findings to a wider population, of which qualitative studies are often not intended or designed to do. Maxwell (2012) describes two types of generalisation: internal - those generalisations made to other groups within the setting; and external – those generalisations made to other settings or individuals (p.142). Throughout the interviews and observations made, the role and influence of the researcher must be considered carefully when analysing the data, as the descriptions and interpretations are based on a short-term relationship, which relies upon relating events to the participants' prior experiences and perspectives. Again, the background data collected on each of the pupils, coupled with the detailed discussions with the head teacher beforehand, sought to minimise this effect within the current study. In considering the complexity of the social world and postpositivist epistemological perspectives, which view meaning as contextually-dependent, as discussed earlier, Bassey (2001) suggests the concept of 'fuzzy generalisations' as a more appropriate way of drawing 'usefulness' from social research findings (Bassey, 2001). Unlike scientific generalisations, which usually suggest that a particular event gives rise to a specific consequence, Bassey's slight change in linguistic use, proposes that fuzzy generalisations are expressed in the form of 'particular events may lead to a particular consequence' (p.6) – those generalisations made are 'true in most situations, but not necessarily all' (p.9). Aligning with QCA, fuzzy generalisations set out to describe the 'conditions under which a particular phenomenon may, or may not, occur' (p.9), which is dependent upon the CMO configurations discussed earlier, and form the basis of any claims made within the current study.

Aligning with 'theoretical generalisations' sometimes used within health care research (Sim & Mcsp, 1998), Bassey (2001) proposes that in educational research, value should be given to the 'relatability' of the research (p.5), where consideration is given to how well teachers can relate such findings to their own classroom practice and experience. He suggests that a more useful output for teachers and educationalists, would be the use of 'fuzzy predictions' (p.12). The study draws on this concept when considering the implications for future practice, discussed in Chapter 6.3).

Whilst there are some challenges to Bassey's (2001) concepts, particularly with respect to the uniqueness and validity of fuzzy generalisations (Hammersley, 2001), Pratt (2003) supports Bassey (2001) in the consideration of the 'usefulness' of generalisations from the practitioner's perspective, or external validity, as being context-dependent, rather than attempting to generalise across contexts (Pratt, 2003, p. 27). He argues in support of Bassey (2001) that for learning to take place, many variables need to be considered, as in the current study, thus the context-dependence of fuzzy generalisations as a guide for usefulness within the classroom, can be a powerful tool. However, despite this agreement with aspects of Bassey's (2001) notion of fuzzy generalisations, Pratt (2003) continues to reinforce the importance of internal validity measures, such as trustworthiness, as a crucial element to conducting and reporting sound research.

Embedded within the current study and with direct reference to QCA, Thomann and Maggetti (2017) argue that to establish inference within this approach, requires 'three intertwined design components': clarification of external validity; ensuring robust internal validity; and 'explicitly adopting a specific mode of reasoning' (p.2). However, it is also essential that theoretical and conceptual arguments are used to support the empirical evidence in order to offer a plausible explanation as to why a particular condition is necessary for an outcome (Schneider & Wagemann, 2013).

Overall, validity of any type, is dependent upon the evidence presented to warrant any claims made. Thus, when assessing the evidence from the current study, it must be remembered that this is context-dependent and is based on 'process theory' (where the focus is on the processes of events, rather than the variables, in order to support any claims) (Maxwell, 2012). Consideration must be given to: how the evidence was gathered; the plausibility of alternative claims; and the context-dependence of the evidence used to warrant particular claims, all of which are transparently presented within the current thesis. As Maxwell (2012) describes, there must be some connection between the fact and the claim, which can be explained – the mechanism. The overall quality of the study, according to Maxwell's (2012) validity typologies from a realist perspective, is thus dependent upon the credibility of the interpretations and conclusions and the consideration of plausible alternative explanations from the evidence.

Chapter 4: Findings

This chapter discusses, in detail, the findings from the current study. Before discussing each of the key findings, at length, a summary of the main findings from the study are provided are provided, to provide the reader with an overview of the aspects discussed within the chapter.

The outcome of interest within the current study is reaching the correct solution to a two-step mathematical word problem. The word problem used, was matched to the age of the pupil through National Curriculum objectives (discussed above). Each of these problems were designed to lend themselves to being solved using the bar model. The outcome measure was a crisp-set measure within the current study, as it has a simple binary outcome: correct solution reached. Further research into this area may be used to explore the methods used in more detail, allowing for the outcome measure to be based on a fuzzy-set, where correct strategies are applied, but the correct solution may not be reached. Reaching the correct solution to the mathematical word problem-solving task, in the current study, must be acknowledged as a temporary outcome, or improvement, based on the temporality of the study. The temporal outcome change within the present study is based on the presence (or absence) of the conditions and activation of any mechanisms at the time of the pupils completing the mathematical word problem-solving task. Furthermore, when considering a critical realist approach to reality, reaching the correct solution to the mathematical word problem-solving task may be considered an empirical representation of the trace of ability to succeed in a maths task, which can be completed using the bar model (the domain of the real). Through applying the recommendations based on the current findings, discussed in detail in chapter 5, such changes, or improvements, may indeed lead to long-term, or structural changes in the brain following implementation of these recommendations over time. However, there is no evidence to warrant such a claim based on the temporal ‘snapshot’ design of the current study, in which performance on the mathematical word problem-solving task (and therefore the outcome) occurred at a fixed point in time and must therefore be acknowledged when interpreting the findings.

Based on the analysis of the qualitative data from the study, the following conditions (or mechanisms) are considered significant for autistic pupils to reach the correct solution to the mathematical word problem:

- Mathematical attainment (ARE, or above).
- Uninhibited executive functions (≤ 4 deficits), specifically attention and working memory.
- Pupils' self-perception of mathematical ability (average, or high).
- Conceptual understanding when using the bar model.

In contrast, the data discussed above indicates little, if any, influence of the following conditions on reaching the correct solution to the mathematical word problem:

- Reading attainment.
- Weak central coherence (WCC).
- Use of the bar model.
- Type of bar model used.

In terms of QCA analysis, three conditions were identified as being sufficient for reaching the correct solution to the mathematical word problem for autistic pupils:

- Not using the bar model.
- High mathematical attainment (≥ARE).
- High levels of pupils' self-perception of their own mathematical ability.

Thus, the minimum formula for sufficiency is:

~BM+*MAtt*SPerc →CorrSoln

That is, *absence of the bar model <u>and</u> high mathematical attainment <u>and</u> high levels of self-perception of mathematical ability are sufficient for reaching the correct solution.*

The chapter now moves on to discuss each of these findings, in detail, using data from the study to exemplify and justify the findings.

4.1 Participant information and case background data

Table 11, below, shows the descriptive statistics of the sample used within the study. All pupils had either a formal diagnosis of autism with no co-morbid conditions (ASD) or no formal diagnosis at all, therefore classed as neurotypical (NT) (discussed in chapter 3.4.1). All schools, included within the data analysis, had been using the bar model within the teaching of mathematics for a minimum of two years (see raw case data in table 11, below) at the time of data collection, as discussed within the PICO criteria (see table 4). This was verified through the discussions with the class teachers.

		Mean ($\bar{x}$)	Range
Age	All cases (N=9)	10 years and 0 months	8 years – 11 years
	Autistic cases (n=7)	9 years and 11 months	8 years – 11 years
	Neurotypical cases (n=2)	10 years and 6 months	10 years – 11 years

Table 11: Descriptive statistics for the cases within the study sample. (The mean values and ranges are presented here to enable any findings to be aligned with previous studies, matched on age)

Data collection for each case consisted of the following steps:

- Discussion with the pupil's class teacher to ascertain pupil attainment data; details pertaining to the behavioural characteristics of the pupil; and information regarding the use of the bar model within the school;
- Interview with the pupil to gain an understanding of their level of interest in mathematics, their self-perception of mathematical ability and their attitudes towards mathematical problem-solving;
- Completion of the mathematical word problem-solving task;
- Further discussion with the pupil regarding their choice of strategy and possible alternative approaches;
- Execution of the STEM sentence completion task.

Following data collection using the instruments discussed in chapter 3.4.3, and the process described above, all raw data was collated and entered into a raw case table (see table 12 below). As discussed earlier, the term 'case' refers to each pupil participating within the study, and is inclusive of, and bounded by, the school context in which it resides. The 'case' data consists of:

- data relating to the school, which the pupil attends;
- background data provided by the class teacher through discussion with the researcher;
- pupil perceptions and behaviours obtained from the interview with each pupil;
- performance on the mathematical problem-solving task by the pupil;
- results from the Stem sentence completion task.

The data obtained from the class teacher discussions provided background information for each of the pupils, as well as information and verification pertaining to the length of time the bar model had been used within the school. Further data was collected throughout these discussions, including more specific behaviours displayed by the pupils (see 'Refinement of Research Design', Chapter 3.5.2) and the extent to which the bar model was embedded or utilised within the teaching and learning of mathematics within the school. This additional data contributed to the qualitative data analysis of each case and was subject to *a priori* and inductive coding within NVivo 12 (QSR International Ltd, 2018) (discussed in chapter 3.6.1).

Interspersed within the pupil discussions, each pupil was presented with a two-step mathematical word problem, matched to the National Curriculum objectives for the appropriate year group (as discussed previously). Scoring for this task, which yielded the outcome measure for the current study, was based on the condition measures discussed in chapter 3.5.1 and was therefore coded as 1 (for reaching the correct solution to the problem) or 0 (for reaching the incorrect solution to the problem). The strategy used by the pupils did not affect the overall scoring for correct completion of this task, as this was analysed separately. Each pupil was left to select their own approach to solving the mathematical word problem initially. Those pupils who did not choose to use the bar model through choice were then asked if the bar model could be used to help solve the problem. Subsequent discussions between the researcher and the pupil explored the mathematical strategies and choices made by the pupil during this task. In the following report, the term 'pupil' is used synonymously with 'case', providing a less dehumanising term.

Finally, the Stem sentence completion task was administered to each pupil, following completion of the mathematical problem-solving task and pupil discussions. The data for this task was recorded and scored as described in Chapter 3.4.

Within table 12 (below) all cases were entered, including those which were later de-selected from the data analysis, as a reminder for the reader. Those cases de-selected, are identified within table 12, along with the rationale for de-selection. The final column in the table denotes whether the correct solution to the mathematical word problem was reached – the outcome measure[13]. Inclusion of the outcome measure enables the reader to understand the relationships between the configurations of independent variables (conditions), and the dependent variable (outcome), which allows for the construction of deterministic models, rather than solely stochastic models, as is the case in many quantitative analysis approaches (Rohwer, 2011).

[13] Within fsQCA analysis (discussed in chapter 4.10) the outcome measure of 'Yes' (correct solution reached) was given a fuzzy-set value of 1 (fully in the set); the outcome measure of 'No' (correct solution not reached) was given a fuzzy-set value of 0 (fully out of the set). Note, there is no crossover point for the outcome measure, or further calibration of this measure, as the outcome is a binary (crisp-set) value.

Name (Pseudonym)	Case Code	School	Year Group	Teacher	Age (Years)	Time in current school (pupil) (Years)	Diagnosis	Attainment: Reading	Attainment: Mathematics	Length of time using bar model (years)	Stem Sentence Completion Score (/20)	Bar model used (initially, or following prompting)	Correct solution reached (Outcome)
Anan	SchAP1	SchA	6	SchACT1	11	6	ASD	Greater depth	ARE	6	20	Yes	Yes
Raoul	SchAP2	SchA	3	SchACT2	8	4	ASD	ARE	<ARE	3	20	Yes	No
Mark	SchGP1	SchG	5	SchGCT1	10	5	ASD	4W (<ARE)	4W (<ARE)	3	14	Yes	No
Eddie	SchBP3	SchB	6	SchBCT1	11	5	ASD	>ARE	>ARE	2	13	No	Yes
Liam	SchBP1	SchB	5	SchBCT2	10	6	ASD	<ARE	ARE	2	15	No	Yes
Andrew^	SchBP4	SchB	6	SchBCT1	11	2	None	ARE	ARE	2	17	Yes	No
Logan^	SchBP2	SchB	5	SchBCT2	10	6	None	ARE	ARE	2	19	No	Yes
James	SchCP1	SchC	5	SchCCT1	10	6	ASD	ARE	>ARE	2	20	Yes	Yes
Joseph	SchCP2	SchC	4	SchCCT2	9	5	ASD	ARE	<ARE	2	10	Yes	No
Henry*	SchFP1	SchF	4	SchFCT1	9	5	ASD	ARE	ARE	<1	16	N/A	N/A
Nathan*	SchFP2	SchF	5	SchFCT2	10	6	ASD	<ARE	<ARE	<1	18	N/A	N/A
Kian*	SchAP3	SchA	2	SchACT3	6	4	ASD	ARE	<ARE	2	0 (NV)	N/A	N/A
Alex*	SchAP4	SchA	2	SchACT3	6	4	ASD	<ARE	<ARE	2	0 (NV)	N/A	N/A

*Cases de-selected: SchAP3/P4 due to pupils being non-verbal; SchF due to using the bar model for <1year

^ Neurotypical pupils (cases)

Table 12: Raw case data

Key
ASD: Autism spectrum disorder
ARE: Age-related expectation
4W: Working at year 4 age-related expectation
NV: Non-verbal

4.2 Reading and mathematical attainment

The measures for reading and mathematics attainment were based on the schools' current pupil tracking data and the most recent assessments available at the time of data collection. All schools within the study, despite potential inconsistencies in the subjectivity of pupil attainment and tracking, utilised similar terminology pertaining to pupil attainment:

- Significantly below age-related expectations (0)
- Working towards age-related expectations (0.3)
- Working at age-related expectations (0.5)
- Working above age-related expectations (0.7)
- Working at greater depth (1)

The numbers in brackets above, denote the calibrated measure for the condition used within QCA analysis (discussed in chapter 3.5.1).

The measures for each case for reading and mathematics attainment can be seen in table 12 above and range from significantly below ARE to those working at greater depth, demonstrating a broad coverage of attainment within the study sample. The two neurotypical cases were working at age-related expectation (ARE) for both reading and mathematics. Of these two cases, one reached the correct solution to the mathematical word problem (discussed below).

Table 13, below, shows the comparison between the reading attainment of each case and whether the correct solution was reached to the mathematical word problem-solving task.

Case	**Reading attainment**	**Correct solution reached**
SchAP1	Greater depth	Yes
SchAP2	ARE	No
SchBP1	<ARE	Yes
SchBP2	ARE	Yes
SchBP3	>ARE	Yes
SchBP4	ARE	No
SchCP1	ARE	Yes
SchCP2	ARE	No
SchGP1	4W (working towards ARE)	No

Table 13: Data comparing reading attainment of each case against the correct solution reached

The data in table 13, above, indicates a broad set of attainment data relating to reading for the participants within the study. Despite the large variation in reading attainment, there appears to be no correlation between this measure and reaching the correct solution to the mathematical word problem. Whilst case SchAP1 was working at greater depth (1) in reading and reached the correct solution, in contrast, case SchBP1 was significantly below age-related expectation in reading, yet still reached the correct solution. Of those cases working at ARE in reading, two reached the correct solution to the mathematical word problem and three did not. For those cases working below ARE (<ARE or 4W (working towards ARE)) (n=2), one reached the correct solution to the mathematical word problem, and one did not.

The data suggests no positive (or negative) correlation between pupils' reading attainment level and reaching the correct solution to the mathematical word problem, therefore suggesting that reading attainment is not a significant condition within reaching the correct solution to the mathematical problem-solving task, based on the sample used within the current study.

In terms of mathematical attainment, the comparisons between this measure and reaching the correct solution to the mathematical problem-solving task are displayed in table 14, below.

Case	Mathematical attainment	Correct solution reached
SchAP1	ARE	Yes
SchAP2	<ARE	No
SchBP1	ARE	Yes
SchBP2	ARE	Yes
SchBP3	>ARE	Yes
SchBP4	ARE	No
SchCP1	>ARE	Yes
SchCP2	<ARE	No
SchGP1	4W (working towards ARE)	No

Table 14: Data comparing mathematical attainment of each case against the correct solution reached

Once again, a broad range of mathematical attainment data was represented within the sample participants. Unlike reading attainment, the data presented in table 14, above, suggests a positive correlation between mathematical attainment and reaching the correct solution to the mathematical word problem. All those cases, whose mathematical attainment was below age-related expectation (<ARE) (n=3), did not reach the correct solution. In contrast, all cases (n=3) who were working above age-related expectation (>ARE) in mathematics, reached the correct solution. Four of the cases were working at ARE in mathematics; of these four, three reached the correct solution to the mathematical word problem. The one case who did not (SchBP4), was a NT case. Consequently, the data indicates that those pupils reaching the correct solution to the mathematical word problem were all working at ARE, or above, in mathematics; the only anomaly in this finding is a neurotypical case. However, it is important to acknowledge at this point that the category of 'working at ARE' still constitutes a potentially broad range of abilities. Although the sample size within the current study is limited, the data available indicates mathematical attainment to be a significant factor in reaching the correct solution to the mathematical word problem, particularly amongst the autistic cases.

4.3 Executive functions

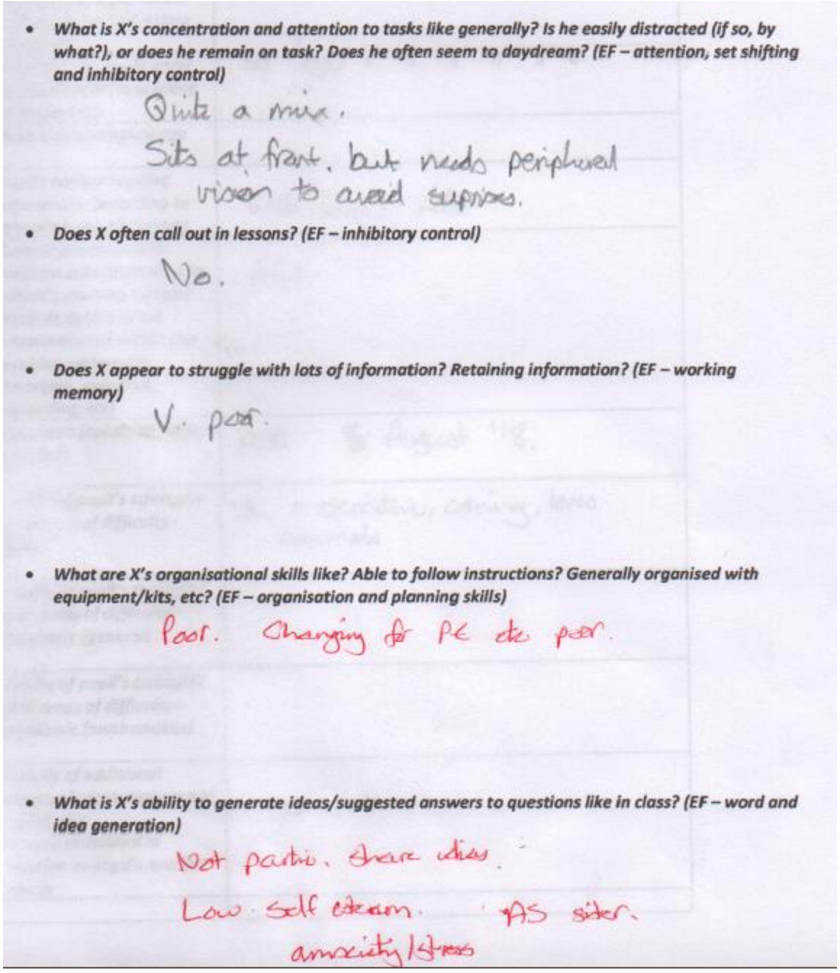
Details pertaining to the executive function (EF) skills and their associated behaviours (when inhibited), were collated from the teacher discussions, through the questions posed in table 15, below.

Question	Associated EF skills
What is X's concentration and attention to tasks like generally? Is he easily distracted (if so, by what?), or does he remain on task? Does he often seem to daydream?	Attention; decision making; inhibitory control
Does X often call out in lessons?	Inhibitory control
Does X appear to struggle with lots of information? Retaining information?	Working memory
What are X's organisational skills like? Able to follow instructions? Generally organised with equipment/kits, etc?	Organisation (and planning) skills
What is X's ability to generate ideas/suggested answers to questions like in class?	Word and idea generation/attention and decision making

Table 15: Questions posed to class teachers to ascertain data on specific behaviours and their associated executive functioning skills

Figure 27, below, provides an example of the responses provided by the teacher for SchBP1, relating to EF skills, to indicate how such data was obtained. The responses indicated that SchBP1 demonstrated behaviours within the classroom indicative of the following inhibited EF skills:

- Attention
- Decision making
- Working memory
- Organisation.



- ***What is X's concentration and attention to tasks like generally? Is he easily distracted (if so, by what?), or does he remain on task? Does he often seem to daydream? (EF – attention, set shifting and inhibitory control)***

Quite a mix.
Sits at front, but needs peripheral vision to avoid surprises.

- ***Does X often call out in lessons? (EF – inhibitory control)***

No.

- ***Does X appear to struggle with lots of information? Retaining information? (EF – working memory)***

V. poor.

- ***What are X's organisational skills like? Able to follow instructions? Generally organised with equipment/kits, etc? (EF – organisation and planning skills)***

Poor. Changing for PE etc poor.

- ***What is X's ability to generate ideas/suggested answers to questions like in class? (EF – word and idea generation)***

Not partic. share ideas.
Low self esteem. AS sister.
anxiety/stress

Figure 27: Responses, relating to behaviours associated with inhibited EF skills, provided by the class teacher for SchBP1

Further information indicating potential inhibition of EF skills (discussed in class teacher discussions above), was also obtained and coded from the pupil interviews. Figure 28, below, demonstrates an example taken from the interview transcript with pupil SchAP2.

R: So, like I said, it's not a test, or anything like that, so it doesn't matter. So,...oh, I'll have to let you use my pencil. So, I'll just explain. On here is a question. You don't need to worry about these bits – this just tells me it's for year 3. Ok? So, this is the question, and if you want me to read it to you, I will. It's up to you.

P2: I don't mind you reading it to me.

R: Ok. Then, in this box, this is where you can do any working out that you need to do. Ok, so however you think you might need to work it out. And then, when you think you've got an answer, then we put it in that box. Alright? Does that sound ok to you?

P2: Yeah.

R: So, the easy bit...you can do the easy bit first, which is do you want to just pop your name there? And you're year 3 aren't you?

P2: Oh, this pencil is heavy!

R: It is isn't it? It's a bit like a pen actually.

(pupil writes his name)

R: Wow! Your handwriting is very neat!

P2: And the date is...?

R: It is the 27th of the second, 19.

P2: It is a pencil though!

R: It is a pencil yeah. You're in, you're year three aren't you?

P2: Mmm.

Figure 28: Extract from pupil interview for SchAP2, indicating inhibited EF skills – particularly attention

In this extract, data was inferred relating to attention and working memory. When the researcher presented the pupil with a mathematical word problem, and subsequently handed him a pencil with which to complete the problem, his attention shifted. It is clear from the extract that pupil SchAP2 switched his attention from the discussion around mathematical problem solving to the focus on the pencil. Despite the researcher attempting to switch the focus away from the pencil following his initial response of, "Oh, this pencil is heavy!", the pupil reverted his attention back to the pencil a short while later, stating, "It is a pencil though!"

Table 16, below, provides a summary of the inhibited EF skills associated with each case.

Case	Attention	Decision making-	Inhibitory control	Working memory	Organisation
SchAP1					
SchAP2			X	X	
SchBP1	X	X		X	X
SchBP2*					
SchBP3	X	X	X	X	X
SchBP4*					
SchCP1	X	X	X	X	X
SchCP2	X	X		X	X
SchGP1	X	X		X	X

Table 16: Summary of the inhibited EF skills for each case (*NT cases)

Table 16 indicates impaired EF skills within all autistic cases (except for SchAP1), aligning with the theory of EF, discussed in chapter 2. Although SchAP1 appears to be an anomaly with regards to impaired EF skills amongst autistic individuals, consideration was given to the potential mechanism of compensation (discussed below). No assumption was therefore made that this pupil did not have any impaired EF skills, simply that none of the behaviours was observed.

The total number of coded items for each of the EF skills was aggregated for each case (based on the class teacher discussions, pupil interviews and performance on the mathematical word problem-solving task) and can be seen in figure 29, below. As discussed above, other than case SchAP1, all autistic cases displayed some behaviours associated with inhibited EF skills. Of those skills, attention and working memory appeared to be significant factors in those cases where the correct solution wasn't reached: evident in cases SchAP2; SchCP2; and SchGP1. However, it must also be noted that in some cases where the correct solution was reached, there was evidence of inhibited attention and working memory (SchBP1 and SchBP3).

In three autistic cases, where EF skills were seen to be inhibited, the correct solution was reached (SchBP1, SchBP3 and SchCP1). As can be seen in figure 29 below, case SchGP1 displayed significant behaviours associated with all the EF skills recorded and did not reach the correct solution to the mathematical word problem. In three cases where there was no indication of inhibited EF skills, the correct solution was reached: both NT cases (SchBP2 and SchBP4) and one autistic case (SchAP1). As discussed above, and in further detail in Chapter 5, it mut be acknowledged that the anomalous case SchAP1 may be indicative of compensating to 'mask' any inhibited EF skills.

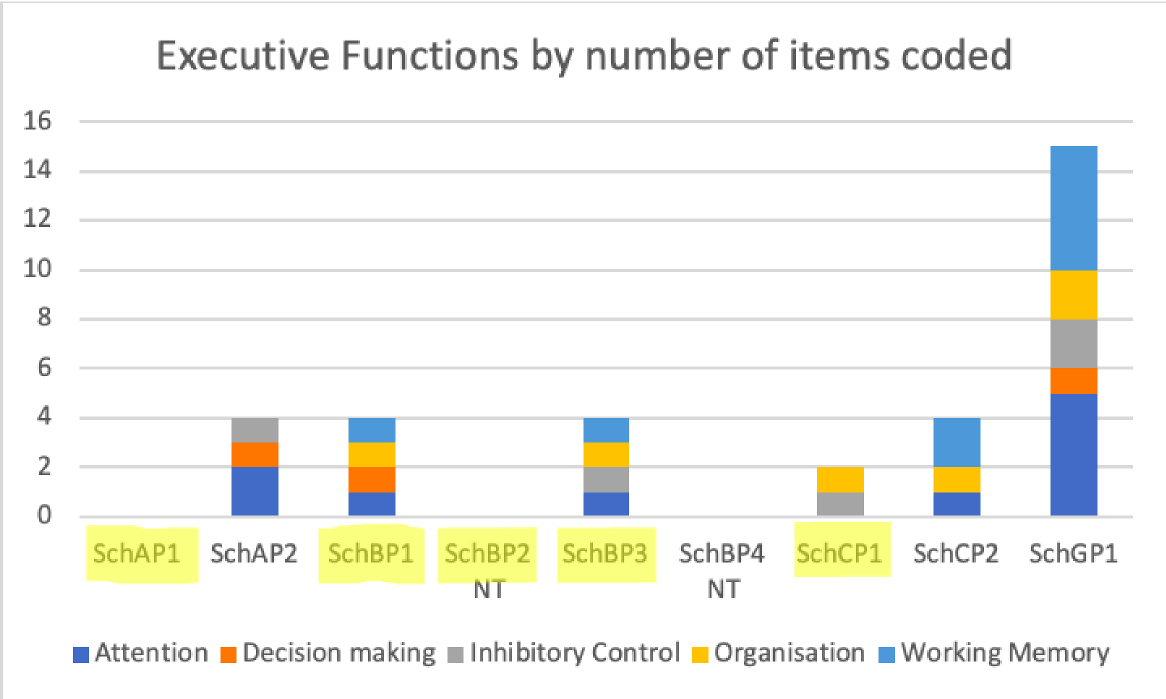


Figure 29: A breakdown of executive functions by the number of items coded per case (highlighted cases indicate the correct solution to the mathematical word problem was reached)

Amongst the autistic cases, the range in the number of coded items relating to inhibited EF skills differed significantly from 0 (SchAP1) to 15 coded items (SchGP1), with a modal number of coded items of 4 (mean 4.7 and median of 4). In contrast, the number of coded items associated with inhibited EF skills for both neurotypical cases was 0. The data suggests therefore, that like autism itself, the profiles of inhibited executive functions within this population, shows large variability, ranging from no observable deficits (or the use of compensation strategies to mask such deficits) through to multiple observable inhibited EFs. In contrast, the data indicates the presence of inhibited EF skills to be associated with the autistic population, rather than the NT population. Furthermore, the data is indicative of a positive correlation between a high number of observable, or recorded, EF deficits (≥4) and failure to reach the correct solution.

4.4 Pupils' self-perception of mathematical ability

Pupils' self-perception of their own mathematical ability was ascertained through the pupil interviews. The measure was based on the Perception of Ability Scale for Students (PASS) (Boersma & Chapman, 1992), as discussed earlier. Pupil responses were coded according to the calibrated measures discussed in chapter 3.5.1 and ranged from 0 (not a high level of self-perceived mathematical ability) to 1 (high level of perceived mathematical ability). The partial set membership score of 0.5 (the crossover point) was allocated to cases where self-perception of mathematical ability was not high, but the pupil was not unhappy with their own mathematical performance.

The transcript below is taken from the interview with SchBP1 and demonstrates how this information was obtained:

R: Ok, so you like pretty much everything in maths then?

P1: Yeah.

R: Well that's good then isn't it? Ok, what about this question then…do you think you're good at maths?

P1: Yeah.

R: You do? And why do you say that?

P1: …I always get every question right.

R: Wow, what more could you ask for! If you get every question right, you must be good. Brilliant! Ok, and…do you do lots of problem solving in maths? You know, if you have like a word problem to solve?

P1: I think we do a lot of that.

R: Ok, you do a lot. I think lots of people do a lot of problem solving. And, so if you're doing problem solving, do you enjoy doing those?

P1: (Nods)

Figure 30: Partial transcript form SchBP1, demonstrating how information pertaining to pupils' self-perception of mathematical ability was obtained

In the exemplar above, for SchBP1, when considering his own self-perception of mathematical ability, his response to the question, "Do you think you're good at maths?" was "Yeah…[because] I always get every question right." Consequently, in terms of quantifying this qualitative data against the calibrated condition measures for use in QCA analysis, a value of 1 was assigned to this response.

In all cases, both autistic and neurotypical, pupils' self-perception of their own mathematical ability appeared to be a significant factor in reaching the correct solution. In all cases, where the correct solution was reached, a self-perception score of 0.5 or 1 (n=5) was recorded. In three cases (SchBP1, SchBP2 and SchBP3), a self-perception score of 1 correlated with reaching the correct solution. Of those cases where the self-perception score was 0.5 (n=4), two of the cases reached the correct solution (both autistic) and two did not reach the correct solution (one autistic and one neurotypical). In all cases where the self-perception score was 0, the correct solution was not reached.

However, when considering pupils' self-perception of their own mathematical ability with their mathematical attainment, consideration must be acknowledged as to the potential interplay between these conditions.

Case	**Mathematical attainment**	**Self-perception of mathematical ability**	**Correct solution reached**
SchAP1	ARE	0.5	Yes
SchAP2	<ARE	0	No
SchBP1	ARE	1	Yes
SchBP2	ARE	1	Yes
SchBP3	>ARE	1	Yes
SchBP4	ARE	0.5	No
SchCP1	>ARE	0.5	Yes
SchCP2	<ARE	0	No
SchGP1	4W (working towards ARE)	0.5	No

Table 17: Data pertaining to mathematical attainment, self-perception of mathematical ability and reaching the correct solution for each case

Table 17, above, indicates some correlation between mathematical attainment and pupils' self-perception of mathematical ability. All cases, whose mathematical attainment is at ARE, or above, indicated average, or high levels of self-perception. In contrast, those cases where mathematical attainment was below ARE, all indicated average, or low levels of self-perception. Furthermore, where levels of self-perception of mathematical ability were high (1), the correct solution was reached in all cases. On the other hand, where levels of self-perception were low (0), the correct solution was not reached.

As discussed further in chapter 5, directionality must be considered here, as it may well have been the case that reaching the correct solution itself promotes high levels of self-perception amongst the pupils and furthermore, a bidirectional relationship may possibly exist between mathematical attainment and self-perception of mathematical ability.

Consequently, the data from the current study suggests a potential three-way relationship between these conditions:

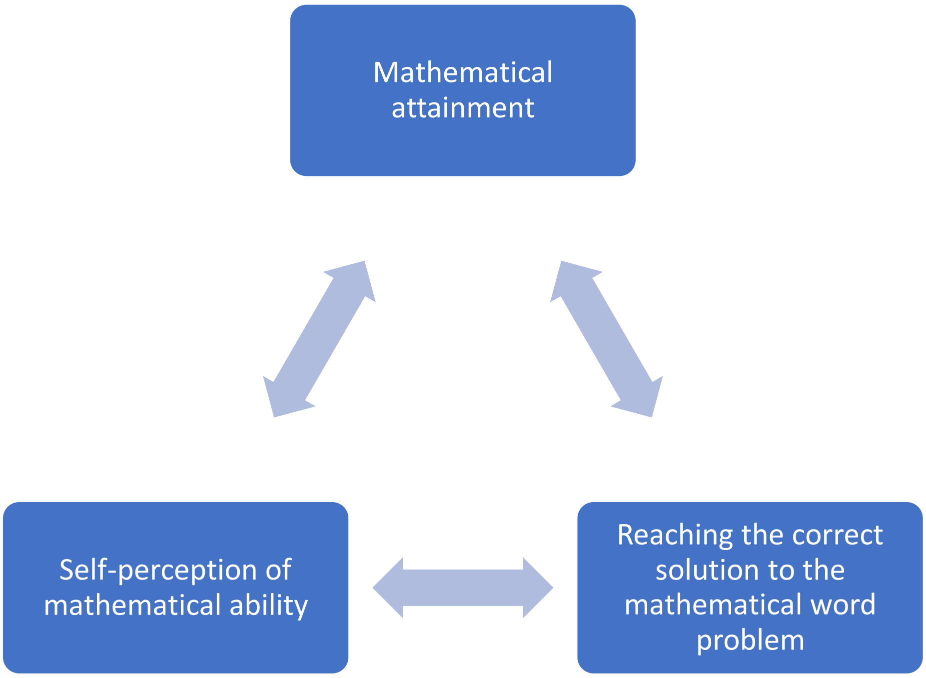
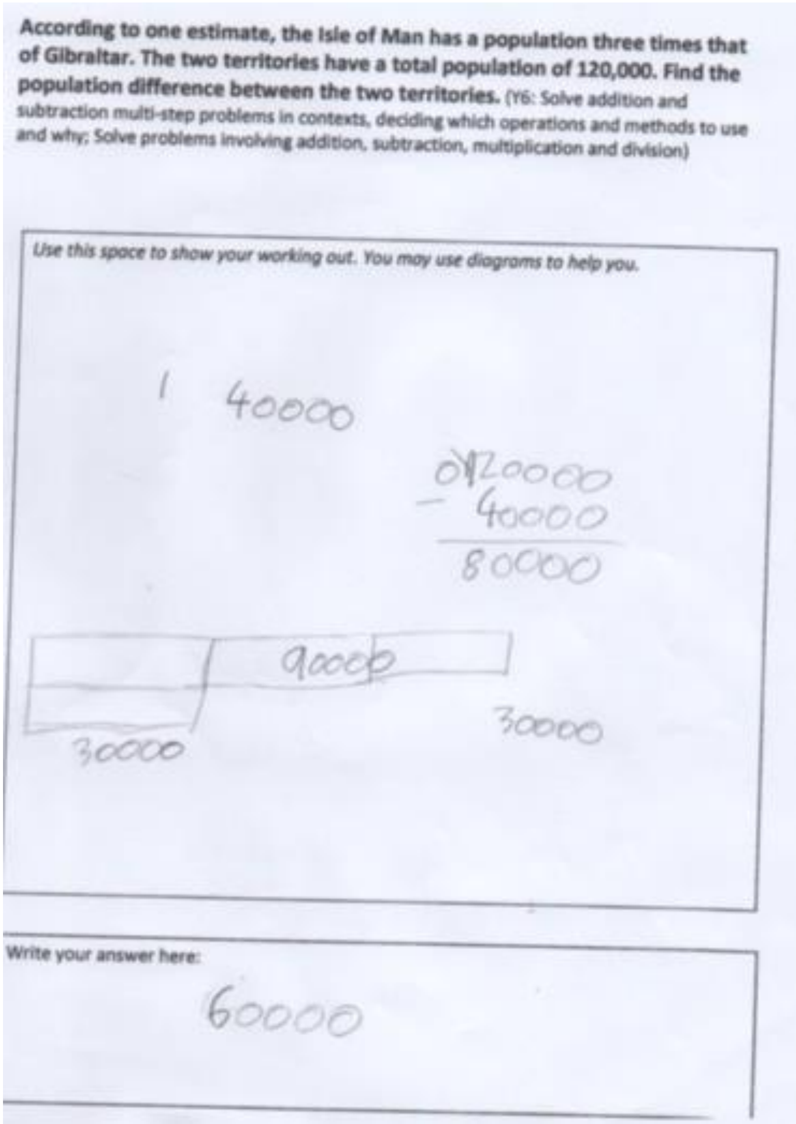


Figure 31: The potential three-way, multidirectional relationship between mathematical attainment, self-perception of mathematical ability and reaching the correct solution

4.5 Reaching the correct solution and use of the bar model

All pupil responses to the mathematical word problem solving task can be seen in appendices xi-xix. Overall, the correct solution to the mathematical word problem was reached by five out of the nine pupils within the study - of these, four were autistic pupils and one was neurotypical. Despite each pupil being exposed to the bar model for a minimum period of two years, only one out of the nine pupils (who was autistic) (SchGP1) chose to use this representation as a strategy for solving the word problem initially. During the interview following the task, when asked if the bar model could be used to support the solution of the mathematical word problem, six out of the nine pupils utilised this approach. Of these six, five were autistic pupils and one was neurotypical.

However, only two of those pupils using the bar model reached the correct solution, both of whom were autistic. Furthermore, of these two pupils, only one represented the correct magnitude and relationships of the numbers within the word problem (SchAP1) when using the bar model (see figure 32 below). This was measured using the visual representation observation form (VROF) discussed in chapter 3.4.3.


According to one estimate, the Isle of Man has a population three times that of Gibraltar. The two territories have a total population of 120,000. Find the population difference between the two territories. (Y6: Solve addition and subtraction multi-step problems in contexts, deciding which operations and methods to use and why; Solve problems involving addition, subtraction, multiplication and division)

Use this space to show your working out. You may use diagrams to help you.

1 40000

120000
− 40000
80000

90000

30000

30000

Write your answer here:

60000

Figure 32: Completion of the mathematical word problem solving task by case SchAP1, showing the correct magnitude and relationships represented within the bar model

Case SchCP1, despite using the bar model and reaching the correct solution, did not represent the magnitude and relationships accurately, possibly suggesting that the bar model was not a significant contributing factor to reaching the correct solution (see figure 33 below). Instead, a combination of estimation, column addition and division were used to generate a likely solution initially. This strategy error was then carried forward to the application of the bar model.

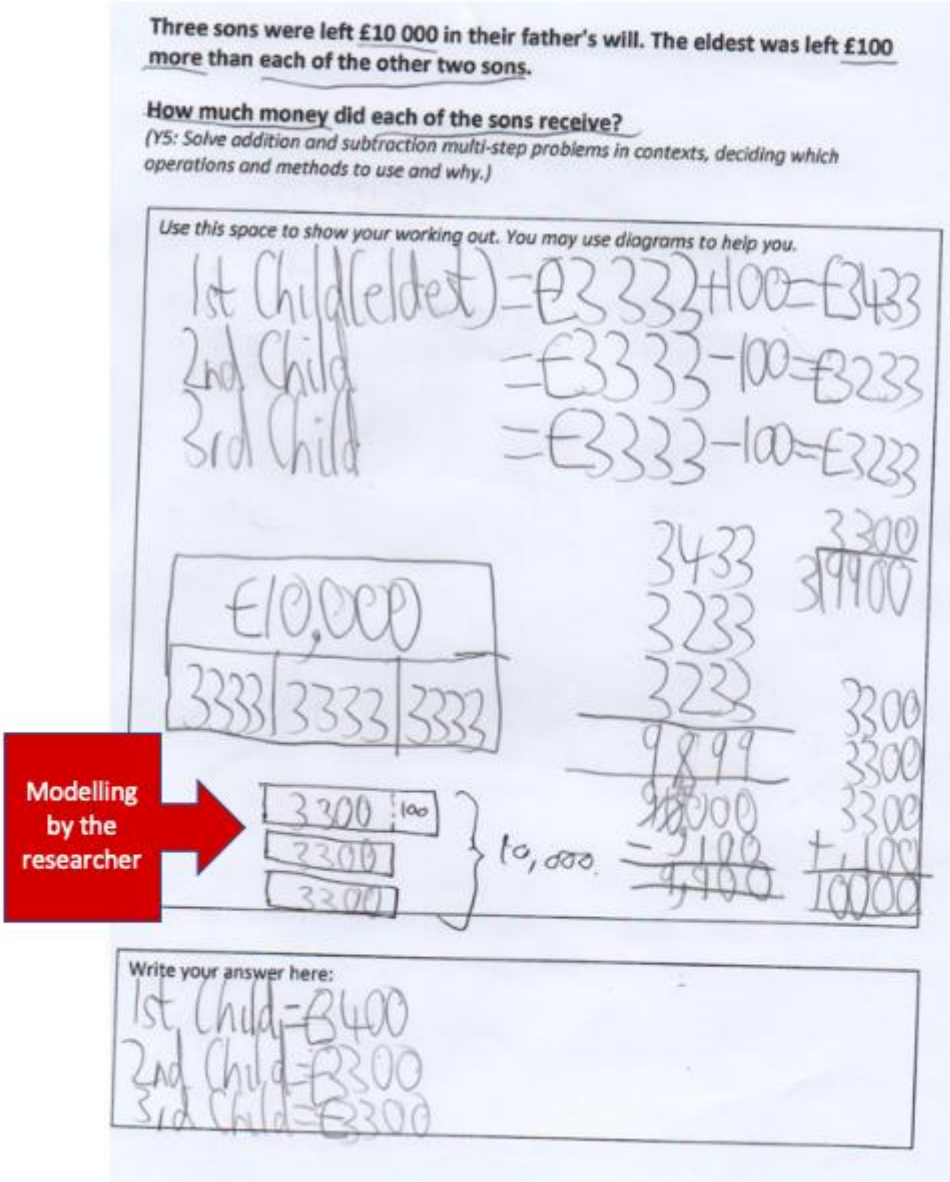


Figure 33: Completion of the mathematical word problem solving task by case SchCP1, showing incorrect magnitude and relationships represented within the bar model, in which the pupil shared the £10,000 equally, before moving onto the misconception of adding £100 to one son and subtracting £100 from the other two sons (the correct representation seen in the example above was modelled by the researcher to SchCP1 following completion of the task)

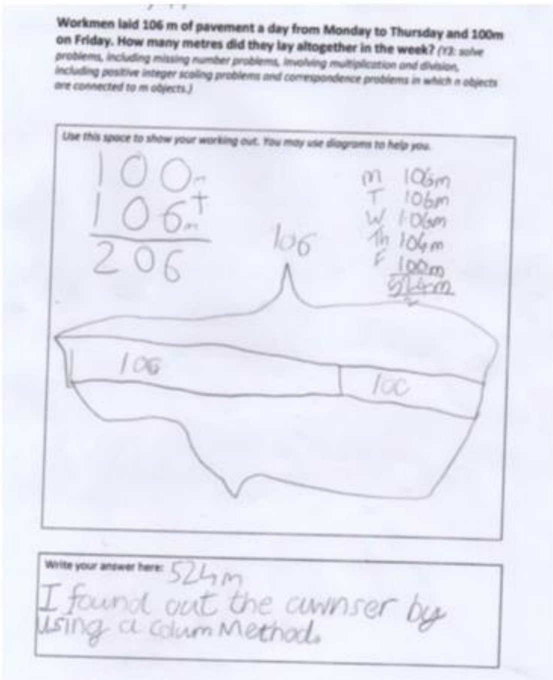
Consequently, the data from the current study provides little evidence in support of:

- Pupils choosing to use the bar model, as a type of visual representation (despite being familiar with it) when solving mathematical word problems;
- Use of the bar model facilitating the correct solution to mathematical word problems.

Nevertheless, the data from the current study is limited and therefore, the above findings, should be treated with caution, as further investigation into this aspect is required, as discussed in chapter 5.

4.6 Procedural and conceptual understanding

The pupils' completion of the mathematical word problem-solving task and subsequent discussion provided some insight into the type of mathematical understanding the pupils drew upon, when it came to application of the bar model, or using alternative methods to solve the word problems. Through analysis of pupils' work and the discussions with the pupils around their approaches to solving the word problems, it was possible to infer potential conceptual and procedural understanding of the pupils. For example, the extract below, from case SchAP2, demonstrates a procedural understanding of the bar model, but a lack of conceptual understanding:



R: Ok. Workmen laid 106m of pavement a day from Monday to Thursday and 100m on Friday. How many metres did they lay altogether in the week?

(pupil begins drawing bar model)

P2: Ok. Is that how long?

R: Yeah, it could be.

P2: And 100 is…that long?

R: Mhm. Ok.

P2: Should I write 106 in here?

R: *(acknowledges)*

P2: And…so…so we don't need to do this…squidgy lines.

R: Ok.

P2: Hmmm. It is 106 and 100 metres…I think I write 106 on top?

R: Ok.

P2: So, what goes at the bottom squidgy line? I don't know. Ok.

R: Don't worry if you're not sure. Because actually, you have drawn a bar model there, haven't you? You've drawn a bar model that matches this calculation *(points to previous attempt to solve problem).*

Figure 34: Working out and extract from the pupil discussion with case SchAP2 around his construction of the bar model (lines 262-291)

In this extract, the pupil was familiar with the process of constructing a bar model, however, demonstrated little, if any, conceptual understanding of how to accurately represent the magnitude and relationships of the numbers presented in the word problem. Should conceptual understanding have been evident, rather than considering each of the magnitudes (100m and 106m) independently, SchAP2 would most likely have considered these to be parts of a whole – an understanding of the part-part-whole concept. Within this concept, the 'whole' relates to the total amount of pavement laid, whereas the parts consist of four parts of 106m and one part of 100m.

Furthermore, through analysing specific responses in the interview transcript above: "squidgy lines" and "I think I write 106 on top?", such references and statements looking for confirmatory acknowledgement, suggests he was drawing on a familiar procedure, in terms of constructing the bar model (procedural understanding), however, was not considering the mathematical concepts embedded within the word problem.

Whether the pupils chose, or attempted, to use the bar model or not, the discussions provided evidence to the researcher as to the type of mathematical understanding the pupils had, based on their selected method.

A further example can be seen from the pupil answer sheet for case SchCP2:

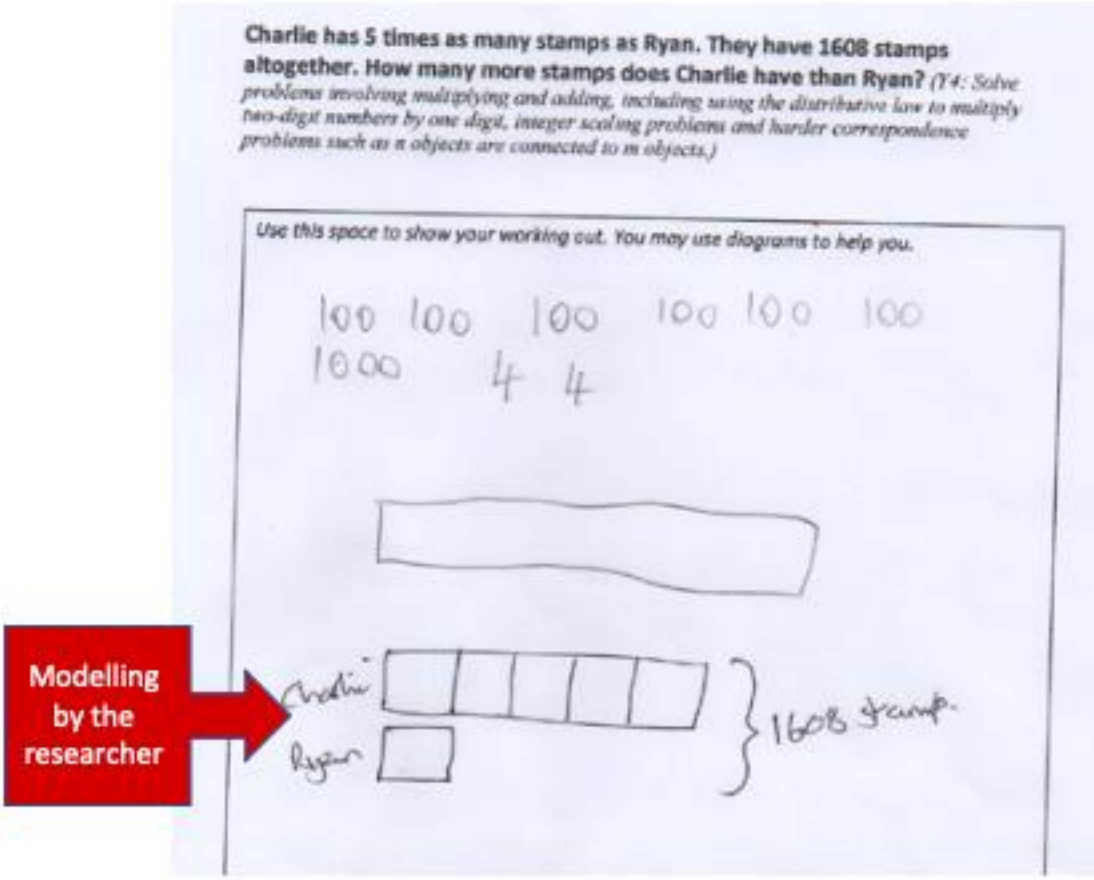


Figure 35: Pupil answer sheet for case SchCP2, showing a lack of conceptual understanding

In the example above, the pupil initially attempted to solve the word problem through counting groups of 100 until reaching the correct solution. When prompted to use the bar model, SchCP2 was able to draw a bar, but had no understanding of how to represent the magnitude and relationships of the numbers from the word problem (the mathematical concepts within the problem), as an accurate bar model. The researcher went on to model the construction of the bar model, however, despite this, SchCP2 was unable to understand the concept represented and consequently failed to reach the correct solution to the word problem.

The mathematical problem-solving task and interview transcript for each pupil was coded, following the procedure discussed above, to identify examples of procedural and conceptual mathematical understanding inferred within each case.

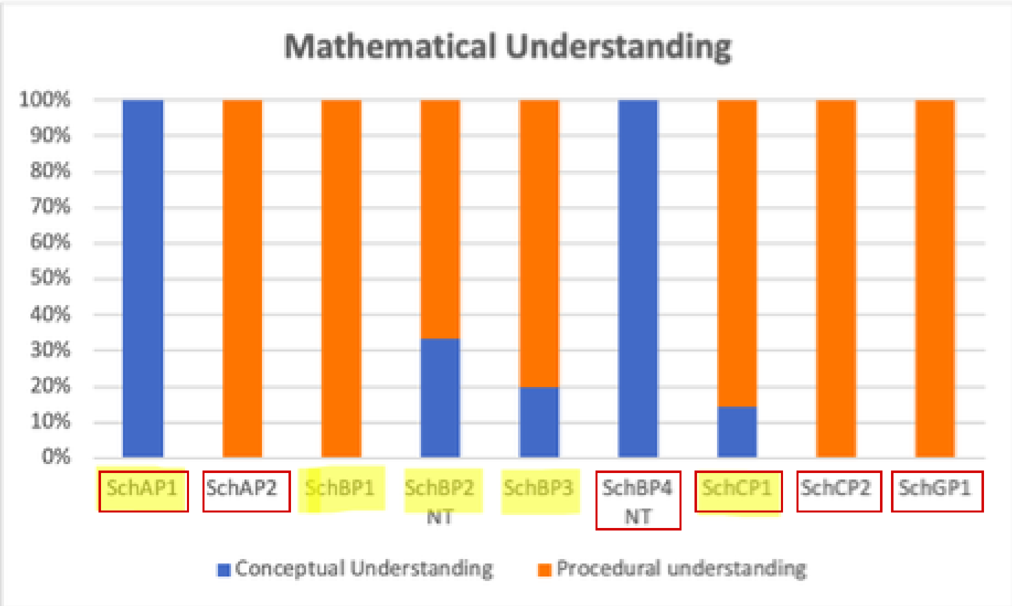


Figure 36: Analysis of mathematical understanding, represented as the % of total items coded within this category for each case (highlighted cases indicate the correct solution to the mathematical word problem was reached; cases in red boxes indicate those who chose to use the bar model)

Figure 36, above, shows the percentage of items coded in NVivo 12 as either procedural, or conceptual understanding for each of the cases within the study. As can be seen, there appeared to be some correlation between the number of items coded as either conceptual or procedural understanding and the ability to reach the correct solution. Of those five cases, where the correct solution was reached, four displayed some conceptual understanding. However, the data also indicates two anomalous results: SchBP1, who reached the correct solution, yet had no recorded conceptual understanding; and SchBP4, who, despite demonstrating some conceptual understanding, failed to reach the correct solution.

Of those cases not choosing to use the bar model (n=4), three of them reached the correct solution: SchBP1, SchBP2, SchBP3. With the exception of SchBP1, the rest of these cases demonstrated some conceptual understanding. Case SchGP1, who did not use the bar model and reached the incorrect solution to the word problem, demonstrated no conceptual understanding within the task.

The cases, who chose to use the bar model, were as follows: SchAP1, SchAP2, SchGP1, SchBP4, SchCP1 and SchCP2. As can be seen from the bar chart above, only those cases, where the correct solution was reached using the bar model, demonstrated conceptual understanding. Such data indicates the requirement to have some conceptual understanding for using the bar model to reach the correct solution to the word problem.

4.7 Type of bar model

Although not specifically selected as a condition for analysis within the application of QCA within the current study (see chapter 3.4.2), the influence of the type of bar model used, on the outcome, must be acknowledged, as this may be indicative of a key condition. As discussed in chapter 2.2.9, the bar model consists of three main variants: the part-part-whole model; the comparison model; and the multiplication-division model (Ciobanu, 2015; Mei & Li, 2014).

Within the current study, two variations of the bar model were used: the part-part-whole model (Year 3 question (n=1) see appendix v) and the comparison model (Year 4, 5 and 6 question (n=8) see appendices vi-viii). Due to the issue of limited diversity, the type of model was not used as a condition for analysis within fsQCA 3.0 (Ragin & Davey, 2016), however, individual analysis of the likely influence of the type of bar model is possible.

Case	Type of bar model	Bar model used (either initially, or following prompting from the researcher)	Correct solution reached
SchAP2	Part-part-whole	Yes	No
SchCP2	Comparison	Yes	No
SchGP1	Comparison	Yes	No
SchBP1	Comparison	No	Yes
SchBP2 (NT)	Comparison	No	Yes
SchCP1	Comparison	Yes	Yes
SchAP1	Comparison	Yes	Yes
SchBP3	Comparison	No	Yes
SchBP4 (NT)	Comparison	Yes	No

Table 18: The type of bar model used and the outcome (correct solution)

Table 18, above, indicates the type of bar model used and the outcome in terms of reaching the correct solution. Only one case (SchAP2) had a mathematical word problem requiring the use of the part-part-whole variation of the bar model. In this case, despite the bar model being used, the pupil reached the incorrect solution to the word problem.

Where the comparison variation of the bar model was used, the outcome was mixed. In only two cases (SchCP1 and SchBP3), both from the ASD subgroup, was the bar model used and the correct solution reached. In three cases (n=2 ASD and n=1 NT), the bar model was used, however the correct solution was not reached. In all cases where the bar model was not used (n=3), the correct solution was reached. Although the current study provides limited data to support the influence of the type of bar model used, the findings from the available data indicate little evidence for the type of bar model being a significant factor (condition) in reaching the correct solution, however, this may be an area for further research (see chapter 6.3).

Nevertheless, further analysis through combining the conditions used within the study, indicates potential relationships (and hence, potential mechanisms). Cases SchCP1 and SchAP1, where the comparison model was used and the correct solution reached, both had a mathematical attainment of above ARE and a minimal number of recorded EF deficits (two and zero respectively), suggesting a stronger influence of these two factors in reaching the correct solution, as discussed above. Furthermore, aligning once again with the findings relating to mathematical attainment and EFs, cases SchCP2 and SchGP1, who both used the bar model but reached the incorrect solution, had a mathematical attainment below ARE and a significant number of recorded EF deficits (4 and 15 respectively), again supporting the findings that these two factors appear significant in reaching the correct solution to the mathematical word problem.

4.8 Stem sentence completion task for assessing weak central coherence (WCC)

The descriptive statistics for the results of the stem sentence completion task are presented in table 19, below. The researcher took the decision to present these statistics using the mean scores and the range, to allow a detailed comparison to those findings in the original study conducted by Booth & Happé (2010).

		Mean ($\bar{x}$)[14]	Range
Score (max. 20)	All cases (N=9)	16.4	10-20
	Autistic cases (n=7)	16	10-20
	Neurotypical cases (n=2)	18	17-19

Table 19: Results, shown as descriptive statistics, for the sentence completion task

Whilst three autistic pupils scored 20 points (maximum score), suggesting no issues with weak central coherence, the next maximum score for this group was 15, indicating more than two local responses. When compared to the normative values suggested by Booth and Happé (2010, p.382), where 23%-32% of participants would score less than 16 (N=217), this suggests weak central coherence to be present in these cases. However, as the score of <16

[14] Although the mean is not used within fuzzy-sets, due to modal measurement (Rohwer, 2011), the stem sentence completion task was used to ascertain indicators of weak central coherence (WCC) within the cases. WCC was not used as a condition for analysis within the present QCA analysis – it was used separately to QCA analysis to ascertain any relationship (correlational relationship) between WCC and reaching the correct outcome (or not). Table 19 therefore uses the mean for the statistics to make a comparison of the statistical findings from the current study, with those from the original stem sentence completion task, carried out by Booth & Happé (2010).

was only present in three out of nine pupils, the figure is close to the expected normative values.

Despite the wide range of scores on the sentence completion task, there appears to be no correlation between these scores (and hence the presence or absence of weak central coherence) and the ability to reach the correct solution to the mathematical word problem. SchAP1 and SchCP1, who both scored 20 on the stem sentence completion task, reached the correct solution. In contrast, SchAP2, who also scored 20 on the stem sentence completion task, did not reach the correct solution. Of those obtaining lower scores on the sentence completion task, SchBP3, with a score of 13; and SchBP1, with a score of 15, both reached the correct solution to the mathematical problem-solving task. Further supporting this lack of apparent correlation between central coherence and mathematical word problem solving, SchCP2, who scored 10, and SchGP1, who scored 14, both failed to reach the correct solution to the mathematical problem-solving task.

4.9 Interim Summary

The findings discussed thus far, indicate the potential significance of the following conditions (or mechanisms) on reaching the correct solution:

- Mathematical attainment (ARE, or above);
- Uninhibited executive functions (≤ 4 deficits), specifically attention and working memory;
- Pupils' self-perception of mathematical ability (average, or high);
- Conceptual understanding when using the bar model.

In contrast, the data discussed above indicates little, if any, influence of the following conditions on reaching the correct solution to the mathematical word problem:

- Reading attainment;
- Weak central coherence (WCC);

- Use of the bar model;
- Type of bar model used.

The following section moves on to consider the findings from the data analysed using QCA. Such analysis enables the identification of configurational pathways indicative of giving rise to reaching the correct solution, in addition to the influence of individual conditions.

4.10 QCA analysis of configurational pathways

Following the data analysis steps discussed in chapter 3.6.2, all raw data was analysed through the use of fsQCA 3.0 software (Ragin & Davey, 2016), in order to ascertain the configurational pathways involved in mathematical problem solving.

Using the raw case data and the calibration measures, in tables 9 and 12, the data was prepared in a table, comparing the source variables with the fuzzy-set values:

ASD									
Reading Attainment	**RAttfz**	**Mathematical Attainment**	**MAttfz**	**Use of the bar model**	**BM+fz**	**Self-Perception of own mathematical ability**	**SPerfz**	**Reaching the correct solution to the word problem (outcome)**	**CorrSolnfz (outcome)**
Greater Depth	1	ARE	0.7	Yes	1	Average	0.5	Yes	1
Greater Depth	1	<ARE	0	Yes	1	Low	0	No	0
4W (<ARE)	0	4W (<ARE)	0	Yes	1	Average	0.5	No	0
>ARE	0.7	>ARE	0.7	No	0	High	1	Yes	1
<ARE	0	ARE	0.5	No	0	High	1	Yes	1
ARE	0.5	>ARE	0.7	Yes	1	Average	0.5	Yes	1
ARE	0.5	<ARE	0.3	Yes	1	Low	0	No	0
NT									
ARE	0.5	ARE	0.5	Yes	1	Average	0.5	No	0
ARE	0.5	ARE	0.5	No	0	High	1	Yes	1

Table 20: Source variables and fuzzy-set (fz) values for the case data (ASD: autistic pupils; NT: neurotypical pupils)

Following this, XY plots were made to explore the sufficiency of each condition. ‘For a condition (X) to be sufficient for the outcome (Y), each case’s membership in the condition (X) must be ≤ its membership in the outcome (Y)’ (Schneider & Wagemann, 2013, p. 68). On the XY plot, this relates to all cases residing above, or on, the diagonal line (Schneider & Wagemann, 2006). The consistency score (for fuzzy-sets) is calculated as a ratio of the sum of each case’s minimum values across the membership scores in X and Y against the aggregated membership values in X, of all cases.

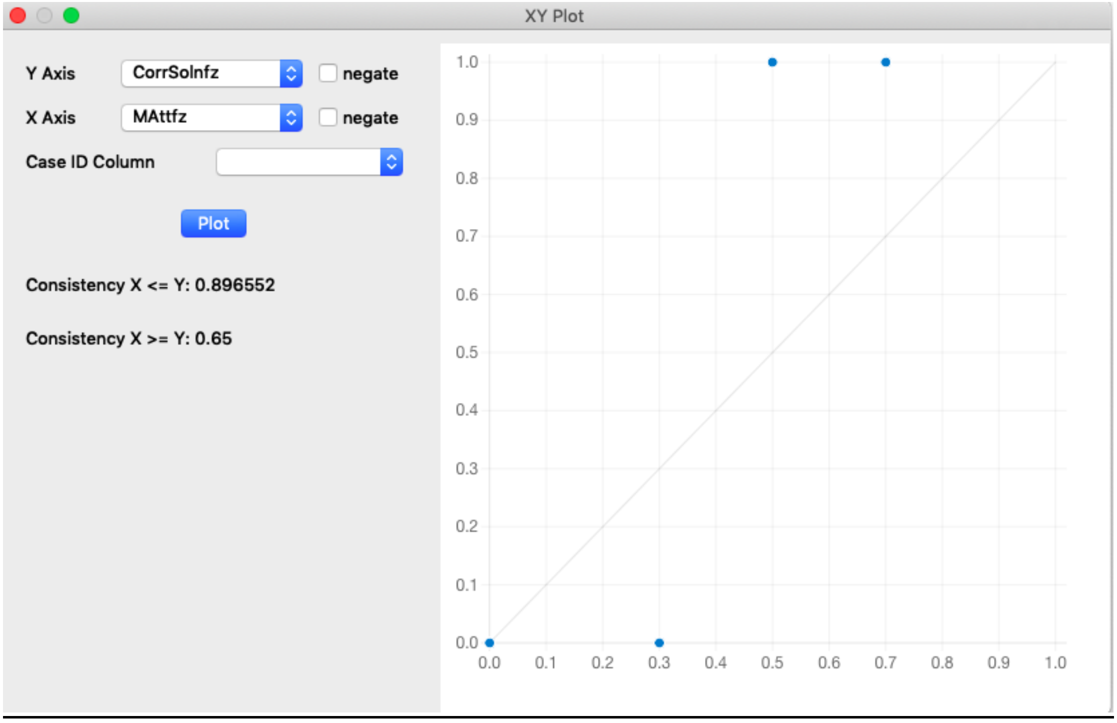


Figure 37: XY plot[15] of mathematical attainment (MAttfz) against the outcome measure (correct solution) for the ASD sub-sample (n=7)

This XY plot in figure 37 above, indicates that the consistency of the fuzzy-set condition of mathematical attainment (MAttfz) being sufficient for the reaching correct solution (CorrSolnfz) is 0.90 (2 d.p.).

The consistency scores of the sufficiency of each condition (X) for the outcome (Y) from each XY plot (discussed in chapter 3.6.2) were then aggregated into tabular form for the data sets pertaining to the ASD subgroup (n=7) and the NT subgroup (n=2) respectively (table 21 and 22 below).

[15] It should be noted here that the line denoted on the XY plots, does not indicate a line of best fit. This is merely the line of X=Y and is automatically generated by fsQCA 3.0 software (C. Ragin & Davey, 2016), which is sometimes helpful for analysis with extremely large data sets.

Condition (X)	Consistency score indicating sufficiency for the outcome (Y)
Reading attainment (RAttfz)	0.594
Not reading attainment (~RAttfz)	0.545
Mathematical attainment (MAttfz)	0.897*
Not mathematical attainment (~MAttfz)	0.341
Using the bar model (BM+fz)	0.4
Not using the bar model (~BM+fz)	1*
Self-perception (SPerfz)	0.857*
Not self-perception (~SPerfz)	0.286

Table 21: The consistency scores for X being sufficient for Y derived from the XY plots, for the ASD sub-sample (n=7)

Condition (X)	Consistency score indicating sufficiency for the outcome (Y)
Reading attainment (RAttfz)	0.5
Not reading attainment (~RAttfz)	0.5
Mathematical attainment (MAttfz)	0.5
Not mathematical attainment (~MAttfz)	0.5
Using the bar model (BM+fz)	0
Not using the bar model (~BM+fz)	1(*)
Self-perception (SPerfz)	0.667
Not self-perception (~SPerfz)	0

Table 22: The consistency scores for X being sufficient for Y, derived from the XY plots, for the NT sub-sample (n=2)

Within table 21 and 22 above, those consistency scores 0.8, or above, are indicated (*). This value was used as the cut-off point, as recommended by researchers in the field (Hirzalla, 2020; Ragin & Davey, 2016). The data above indicates the following conditions to be sufficient in terms of reaching the correct solution to the mathematical word problem:

- High mathematical attainment (MAttfz)
- Absence of the bar model (~BM+fz)

- High self-perception of mathematical ability (SPerfz)

At this point, the limited sample size of the neurotypical group (n=2) yields unreliable data. This is due to limited diversity – where there is insufficient empirical data to support the number logically possible combinations of conditions used (Schneider & Wagemann, 2010), resulting in logical remainders in the truth table – i.e., where there are no cases to support a particular configuration (see data analysis). Consequently, analysis beyond this point only focuses on the data from the autistic sub-group (n=7).

Three causal conditions were identified as sufficient for the outcome: (MAttfz, ~BM+fz and SPerfz). Using these, the subset/superset algorithm within fsQCA 3.0 (Ragin & Davey, 2016) was used to analyse the consistency of the likely configurational pathways of these three conditions, in giving rise to the outcome (the correct solution).

Configuration	Consistency	Coverage (of outcome by condition(s))
MAttfz x ~BM+fz x SPerfz	1	0.30
MAttfz x SPerfz	1	0.55
MAttfz x ~BM+fz	1	0.30
~BM+fz x SPerfz	1	0.50
~BM+fz	1	0.50
MAttfz	0.90	0.65
SPerfz	0.86	0.75

Figure 38: Subset/superset analysis of the three causal conditions, which indicate consistent subset relations to the outcome. Note, the coverage indicated 'expresses how much of the outcome (Y) is covered by condition X, thus expressing the empirical importance of X for explaining Y (Schneider & Wagemann, 2013, p.128).

Using the cut-off point of 0.8, as discussed above, this algorithm suggests that each of the conditions, either solely, in combination with one other, or the combination of all three, are likely to provide a configurational pathway, which gives rise to the correct solution (a sufficient pathway).

The detailed analysis, based on the three conditions (~BM+, MAtt, SPerc), which are subsets of the outcome (correct solution) and based on consistency scores of 0.8, or above, are represented in the following data matrix, which was constructed by extracting all measures from the case data based upon the calibration and recalibration of condition measures discussed in chapter 3.5.1.

Case	**Case Code**		**Choose to use bar model (prompt)** **A**	**Current level of mathematical attainment** **B**	**Pupil's self-perception** **C**	**Correct Solution** **Y**
1	**SchAP1**		*1*	*0.7*	*0.5*	*1*
2	**SchAP2**		*1*	*0*	*0*	*0*
3	**SchGP1**		*1*	*0*	*0.5*	*0*
4	**SchBP1**		*0*	*0.5*	*1*	*1*
5	**SchBP3**		*0*	*1*	*1*	*1*
6	**SchCP1**		*1*	*0.7*	*0.5*	*1*
7	**SchCP2**		*1*	*0.3*	*0*	*0*

Table 23: Data matrix for all cases included in the QCA analysis, based on the three conditions (A, B and C), deemed subsets of the outcome (Y). Note that de-selected cases are no longer included in the analysis at this stage

The data matrix (table 23, above) was simplified to create a fuzzy data matrix, displaying only the data required for the subsequent analysis, using fsQCA 3.0 (Ragin & Davey, 2016). Note here, that as the causal condition for consistency is ***<u>not</u>*** using the bar model (~BM+), the following data tables use the negated values from those in table 23 above (which show the values for using the bar model (BM+).

Case	A (~BM+fz)	B (MAttfz)	C (SPercfz)	Outcome (CorrSoln)
1	0	0.7	0.5	1
2	0	0	0	0
3	0	0	0.5	0
4	1	0.5	1	1
5	1	1	1	1
6	0	0.7	0.5	1
7	0	0.3	0	0

Key
~BM+: Not using the bar model
MAtt: Mathematical attainment
SPerc: Pupils' self-perception of their own mathematical ability

Table 24: Fuzzy data matrix indicating all possible configurations of the three conditions

All possible configurations of conditions (presence = 1, absence = 0) are expressed in table 25 (below).

A (~BM+)	B (MAtt)	C (SPerc)
0	1	1
0	0	1
0	1	0
0	0	0
1	1	0
1	0	0
1	0	1
1	1	1

Table 25: All possible configurations of the three conditions

The membership scores of all cases in all configurations is presented below. A case's membership score within a particular configuration is determined by its lowest membership score in each of the individual conditions. The membership score in a negated condition (absence of the condition) is calculated by subtracting its score when present, from 1. Within the following tables, negated conditions (absence) are indicated by a lower-case letter, whereas their presence is indicated by a capital letter. For example, the configuration in the first row of table 25 above (0,1,1), would be written as aBC, whereas the configuration for the second row of table 25 above (0,0,1), would be written as abC, and so on. Cases are assigned to the configuration in which they have a membership score of >0.5 (indicated in yellow in table 26 below).

Case	A (~BM+)	B (MAtt)	C (SPerf)	Outcome		ABC	AbC	ABc	Abc	aBc	abc	abC	aBC
1	0	0.7	0.5	1		0	0	0	0	0.5	0.3	0.3	0.5
2	0	0	0	0		0	0	0	0	0	1	0	0
3	0	0	0.5	0		0	0.5	0	0	0.5	0.5	0.5	0
4	1	0.5	1	1		0.5	0.5	0.5	0	0	0	0	0
5	1	1	1	1		1	0	0	0	0	0	0	0
6	0	0.7	0.5	1		0	0	0	0	0.5	0.3	0.3	0.5
7	0	0.3	0	0		0	0	0	0	0.3	0.7	0	0

Table 26: Membership of all cases in all configurations (the cells highlighted in yellow have membership scores >0.5 within the configuration)

Hence, the following cases can be assigned to the following configurations (table 27 below):

A (~BM+)	B (MAtt)	C (SPerc)	Cases
0	1	1	0
0	0	1	0
0	1	0	0
0	0	0	2 (Case 2,7)
1	1	0	0
1	0	0	0
1	0	1	0
1	1	1	1 (Case 5)

Table 27: The assignment of cases to configurations where the membership score is >0.5

The data thus far indicated that the configurations abc (0,0,0) and ABC (1,1,1) contain cases with membership scores of >0.5. The configuration abc (0,0,0) contains two cases (case 2 and case 7), and the configuration ABC (1,1,1) contains one case (case 5). These configurations are referred to as non-remainder configurations, as they contain cases with membership scores of >0.5.

To calculate the consistency, the lowest scores for each of the cases in the configuration were used (indicated in yellow cells). Therefore, for the configuration 0,0,0 (abc), the following scores are used:

Case	A (~BM+)	B (MAtt)	C (SPerc)	Outcome (Y)	abc
1	0	0.7	0.5	1	0.3
2	0	0	0	0	1
3	0	0	0.5	0	0.5
4	1	0.5	1	1	0
5	1	1	1	1	0
6	0	0.7	0.5	1	0.3
7	0	0.3	0	0	0.7

Table 28: Membership scores of all cases in configuration 0,0,0 (abc)

Using the formula for consistency[16] of sufficiency for Y (the outcome), discussed in chapter 3.6.2 above, the sum of the lowest scores, when comparing the outcome with the lowest score in the configuration, = (0.3+0+0+0+0+0.3+0)= 0.6 → numerator in the consistency formula. The summed membership scores in the combination (0.3+1+0.5+0+0+0.3+0.7) = 2.8 → denominator, hence the raw consistency was 0.6/2.8=0.214.

This process was then repeated for the configuration 1,1,1 (ABC):

Case	A (BM+)	B (MAtt)	C (SPerc)	Outcome (Y)	ABC
1	1	0.7	0.5	1	0.5
2	1	0	0	0	0
3	1	0	0.5	0	0
4	0	0.5	1	1	0
5	0	1	1	1	0
6	1	0.7	0.5	1	0.5
7	1	0.3	0	0	0

Table 29: Membership scores of all cases in configuration 0,1,1 (ABC)

Again, the consistency was calculated as above. The sum of the lowest scores, when comparing the outcome with the lowest score in the configuration = (0.5+0+0+0+0+0.5+0) = 1.0 → numerator in the consistency formula. The summed membership scores in the combination (0.5+0+0+0+0+1+0) = 1.0 → denominator, hence the raw consistency was 1.0/1.0 = 1.

Using the consistency scores generated above, the following truth table was constructed:

---

[16] Consistency $_{\text{Sufficient conditions } (Xi \leq Yi)}$ = $\dfrac{\sum_{i=1}^{1} \boldsymbol{min(Xi, Yi)}}{\sum_{i=1}^{1} \boldsymbol{Xi}}$

(X refers to the membership score of the condition, Y refers to the membership score of the outcome) (Schneider & Wagemann, 2013, p.126)

A (~BM+)	B (MAtt)	C (SPerc)	Cases	Raw Consistency
0	1	1	0	0
0	0	1	0	0
0	1	0	0	0
0	0	0	2 (Case 2,7)	0.214
1	1	0	0	0
1	0	0	0	0
1	0	1	0	0
1	1	1	1 (Case 5)	1

Table 30: The truth table for the data, based on the consistency scores. For configurations where no cases are present, the raw consistency is 0. Note here, the term 'raw consistency' refers to the 'consistency of a single truth table row' (Schneider & Wagemann, 2013, p.332).

The truth table rows with at least 0.8 raw consistency were given an outcome value of 1 (indicating sufficiency). A minimum value of 0.75 is recommended for the raw consistency score cut off in order to add to the robustness of the findings (Schneider & Wagemann, 2010, p.10). However, as there is no 'universally accepted consistency threshold', the choice of threshold used for the consistency score should be justified by the researcher (Schneider & Wagemann, 2010, p. 10; Schneider & Wagemann, 2013, pp. 127–128). Such features, which may influence the choice of consistency threshold used include as it is research-specific and may vary depending on various factors, such as the number of cases, the context, the research aims and the knowledge of the researcher (ibid.). According to Schneider & Wagemann (2013), a higher threshold should be used where 'the number of cases is low, the number of logically contradictory cases is higher and precision in the calibration process is less confident' (p.128). Hence the cut off used within the current study of 0.8, indicating that any findings have a high consistency across cases within the study. Truth table rows with a raw consistency <0.8 are given a value of 0 (indicating insufficiency). Logical remainders occur due to limited diversity of reality: the cases, and reality generally, are not fully diversified in terms of the conditions analysed (Hirzalla, 2020). Remainders do not get an outcome value of 0 or 1, but code R. Hence, the final truth table is as follows:

A (~BM+)	B (MAtt)	C (SPerc)	Cases	Outcome
0	1	1	0	R
0	0	1	0	R
0	1	0	0	R
0	0	0	2 (Case 2,7)	0
1	1	0	0	R
1	0	0	0	R
1	0	1	0	R
1	1	1	1 (Case 5)	1

Table 31: The final truth table indicating sufficient configurations (with an outcome of 1) and logical remainders, R, where no cases are present

Logical minimisation, through the removal of redundant conditions was not required, as there is only one pathway indicating an outcome of 1.

Therefore, minimum formula for sufficiency = ~BM+*MAtt*SPerc →CorrSoln

The truth table above indicates a consistency score of >0.8 for the configurational pathway MAttfz x ~BMfz+ x SPercfz, thus enabling the outcome measure (CorrSolnfz) to be 1. In contrast, the configurational pathway ~MAttfz x BM+fz x ~SPercfz yields a consistency score of <0.8 (0.214), thus indicating an outcome measure of 0. Consequently, the data suggests that high mathematical attainment (ARE, or above) and high pupils self-perception of mathematical ability (0.5, or above) together form a sufficient pathway to reaching the correct solution, without the requirement of the bar model.

Truth table analysis of the above data was then carried out to establish both sufficient and necessary conditions. The intermediate solutions were used for the following analysis, as discussed earlier, unless stated. The intermediate solution (for the autistic sub-group) for the configuration: MAttfz x ~BM+fz x SPerfz, yielded a consistency[17] score of 1 and a coverage[18] of 0.3.

[17] Consistency indicates the 'degree the empirical data are in line with a postulated subset relation' (Schneider & Wagemann, 2013, p.324).

[18] Coverage expresses how much of the outcome (Y) is covered by condition X, thus expressing the empirical importance of X for explaining Y (Schneider & Wagemann, 2013, p.128).

The intermediate solution above, indicates that based on the data from the current study, the following configurational pathway, is sufficient for reaching the correct solution, in autistic cases:

- High mathematical attainment ***and*** not using the bar model ***and*** high self-perception of mathematical ability.

However, the above result should be treated with caution due to the small-N (n=7) involved within the analysis. Nevertheless, according to Schneider and Wagemann (2013), a test for robustness within the solution terms, is through a similar consistency and coverage score for each term. Although the current study utilises the intermediate solution in terms of analysis (discussed earlier), the two other solution terms were also calculated to demonstrate the robustness of the findings: the complex solution yielded a consistency score of 1 and a coverage of 0.3; the parsimonious solution yielded a consistency score of 0.81 and a coverage of 0.85.

As described above, the solution scores for the intermediate solution are the same as those for the complex solution, indicating robustness in the consistent findings. The parsimonious solution indicates some variation, however, within this solution the plausibility of logical remainders is not considered.

Finally, analysis of necessary conditions was carried out and is displayed in table 32, below. The notation used is as above, with ~ denoting the absence of a condition (negated) and x denoting 'logical and'.

Cases Sample	Fuzzy-set condition	Interpretation	Consistency
Autistic (n=7)	SPercfz	High self-perception of mathematical ability	0.700
	RAttfz	High reading attainment	0.500
	MAttfz	High mathematical attainment	0.580
	BM+fz	Use of the bar model	0.00
	~BM+fz	Not using the bar model	0.600
	BM+fz x SPercfz	Using the bar model ***and*** high self-perception of mathematical ability	1.000
	BM+fz x ~SPercfz	Using the bar model ***and*** not high self-perception of mathematical ability	0.600
	BM+fz x RAttfz x MAttfz x SPercfz	Using the bar model ***and*** high reading attainment ***and*** high mathematical attainment ***and*** high self-perception of mathematical ability	1.000
	~BM+fz x RAttfz x MAttfz x SPercfz	Not using the bar model ***and*** high reading attainment ***and*** high mathematical attainment ***and*** high self-perception of mathematical ability	0.840
	BM+fz x ~RAttfz x ~MAttfz x SPercfz	Using the bar model ***and*** not high reading attainment ***and*** not high mathematical attainment ***and*** high self-perception of mathematical ability	1.000
	BM+fz x RAttfz x MAttfz x ~SPercfz	Using the bar model ***and*** high reading attainment ***and*** high mathematical attainment ***and*** not high self-perception of mathematical ability	0.900
	BM+fz x ~RAttfz x MAttfz x ~SPercfz	Using the bar model ***and*** not high reading attainment ***and*** high mathematical attainment ***and*** not high self-perception of mathematical ability	1.000

Table 32: Analysis of necessary conditions of autistic case data (n=7) with consistency scores

The analysis of necessary conditions was analysed with the consistency cut-off of 0.8, as discussed above. The data in the above table suggests that, for autistic cases, no single condition is necessary for reaching the correct solution. However, when the bar model is used, in the presence of the following conditions, the correct solution is reached:

- High self-perception of mathematical ability
- High reading attainment *and* high mathematical attainment *and* high self-perception of mathematical ability

- High reading attainment *and* high mathematical attainment
- High mathematical attainment

As no single condition always needs to be present with the bar model, to reach the correct solution, it can be established that there is no necessary pathway or condition (i.e. no single condition, or combination of conditions, is a superset of the outcome). However, there are multiple configurational pathways, which are subsets of the outcome, and therefore sufficient for reaching the correct solution. Those sufficient pathways, when using the bar model, are indicated above. In addition to these, for autistic pupils not using the bar model, the following configurational pathway is also sufficient:

- High reading attainment *and* high mathematical attainment *and* high self-perception of mathematical ability.

As reading attainment (RAttfz) results appears to be part of a sufficient pathway at this stage, but not in the analysis above, this can be considered a redundant condition. Its presence or absence (negated) has the same outcome, thus the minimum formula for sufficiency (discussed above) remains as:

~BM+*MAtt*SPerc →CorrSoln

That is, *absence of the bar model and high mathematical attainment and high levels of self-perception of mathematical ability are sufficient for reaching the correct solution.*

In all cases, the data from the current study indicates that it is necessary for the bar model to be used in the presence of other conditions, to form a sufficient configurational pathway (i.e. on its own, it is not sufficient (or necessary) for reaching the correct solution). Consequently, in autistic cases, the bar model requires either high self-perception of mathematical ability, high reading attainment, high mathematical attainment, or a combination of the three, to reach the correct solution (sufficiency). However, in the absence of the bar model, all three conditions are required to be present, to reach the correct solution (sufficiency).

4.11 Manual configurational pathway analysis

As discussed above in Chapter 3.6.2, a manual analysis of the configurational pathways, was also carried out in addition to analysis using fsQCA 3.0 software (Ragin & Davey, 2016). The purpose of this manual analysis was to provide the researcher with a set of comparative configurational pathways derived from the empirical data, to analyse alongside those generated using the software. However, as discussed in chapter 3.6.2, the manual analysis does not allow for any configurations to be analysed, where empirical data is not available.

The raw case data, along with the source variable and fuzzy-set values were used to analyse the configurational pathways giving rise to the absence or presence of the correct solution for each case, based solely on the empirical data collected. Manual analysis yielded the following configurational pathways (note the notation used for these pathways is the same as that used within fsQCA 3.0 software (Ragin & Davey, 2016) analysis. Negated conditions are represented by ~, and 'logical and' is represented by x). Autistic cases and neurotypical cases are identified using ASD and NT respectively. Any condition measures on the crossover point, have been identified as present in the following analysis (e.g. RAttfz and MAttfz measures of 0.5 are taken as present, as they indicate the pupil is working at age-related expectation):

1. All correct solution pathways:
    a. BM+fz x RAttfz x MAttfz x SPercfz (ASD)
    b. ~BM+fz x ~RAttfz x MAttfz x SPercfz (ASD)
    c. ~BM+fz x RAttfz x MAttfz x SPercfz (ASD)
    d. ~BM+fz x RAttfz x MAttfz x SPercfz (NT)
    e. BM+fz x RAttfz x MAttfz x SPercfz (ASD)
2. Correct solution pathways <u>using the bar model</u>:
    a. BM+fz x RAttfz x MAttfz x SPercfz (ASD)
    b. BM+fz x RAttfz x MAttfz x SPercfz (ASD)
3. Correct solution pathways <u>not using the bar model</u>:
    a. ~BM+fz x ~RAttfz x MAttfz x SPercfz (ASD)
    b. ~BM+fz x RAttfz x MAttfz x SPercfz (ASD)

c. ~BM+fz x RAttfz x MAttfz x SPercfz (NT)

4. Incorrect solution pathways:
    a. BM+fz x RAttfz x ~MAttfz x ~SPercfz (ASD)
    b. BM+fz x RAttfz x ~MAttfz x ~SPercfz (ASD)
    c. BM+fz x ~RAttfz x ~MAttfz x SPercfz (ASD)
    d. BM+fz x RAttfz x MAttfz x SPercfz (NT)

In the above analysis, it can be inferred that both RAtt+ and BM+ are redundant conditions (not required in the final solution pathway, as both their presence and absence, gives rise the outcome (reaching the correct solution). As with the fsQCA 3.0 (Ragin & Davey, 2016) analysis, the manual pathway analysis indicates that self-perception of mathematical ability is required to be high (or at least average – the crossover point) in order to reach the correct solution. Furthermore, attainment in mathematics is also indicated to be required to be at least working at age-related expectation, to reach the correct solution. As with the fsQCA (Ragin & Davey, 2016) analysis, discussed above, reading attainment appears to be less of a consistent condition influencing the correct solution pathway. Also aligning with the fsQCA 3.0 (Ragin & Davey, 2016) analysis above, the manual analysis of the configurational pathways suggests that where the bar model is used, additional conditions are required in order to reach the correct solution – in the manual analysis, this indicates the presence of high reading attainment, high mathematics attainment and high self-perception of mathematical ability. Unlike with fsQCA analysis, additional combinations of these conditions is not possible when manually analysing the pathways, as only the empirical data available can be used.

Through comparing the results from fsQCA 3.0 analysis and the configurational pathways through manual analysis, there is significant alignment between the findings. This therefore suggests that the use of fsQCA 3.0 (Ragin & Davey, 2016) software provides reliable output data, based on the small-N data set used within the current study.

4.12 Summary of key findings

This section has discussed in detail the findings from each of the data collection instruments used within the main study. The data discussed, suggests no correlation between pupils' reading attainment and reaching the correct solution to a mathematical word problem in both ASD and NT cases (based on N=9). However, in terms of mathematical attainment, the data available indicates that only those pupils working at age-related expectation (ARE) or above, are likely to reach the correct solution. Consequently, mathematical attainment may be a significant factor (condition) required for reaching the correct solution for all pupils (ASD and NT), according to the data within this study (N=9).

Unsurprisingly, all autistic cases, except for SchAP1, displayed evidence of some inhibited EF skills (discussed in chapter 2.1). Of all the EF skills considered, both attention and working memory appear to indicate some significance in reaching the correct solution. In 3 out of 5 autistic cases, inhibition of these EF skills correlated with the correct solution not being reached, suggesting that attention and working memory may be key factors (conditions) required for reaching the correct solution in autistic cases.

Unlike EF skills, WCC (discussed in chapter 2.1) appears to show no significance on reaching the correct solution. The variation in the stem-sentence completion scores (used to indicate the presence of WCC) bears no correlation to reaching the correct solution to the mathematical word problem for the cases within the current study.

The data from the current study indicates pupils' self-perception of their own mathematical ability appears to be a significant factor (condition) for both ASD and NT cases, in reaching the correct solution. Only cases where pupils' self-perception of mathematical ability was recorded as 0.5 (partial set membership) or 1 (full set membership) was the correct solution reached. Nevertheless, what the current study does not establish, is whether any relationship between high levels of self-perception of mathematical ability and high levels of attainment in mathematics exist. One of these conditions may indeed be a driver for the other – an area for potential future research (see chapter 6.3).

Despite the bar model being incorporated into the teaching of mathematics in all schools within the study, for a minimum of 2 years, this visual representation was rarely chosen to support the solution to the mathematical word problem. When prompted to use this approach, 6 out of the 9 pupils made use of it (compared to only 1 before prompting). However, only 2 of these 6 pupils reached the correct solution when using the bar model, both of whom were autistic cases. The ability to accurately represent the magnitude and relation of the numbers in the word problems, through the bar model, appeared to be a significant issue in most cases. Consequently, the data from the current study suggests that the bar model alone may not be a significant factor (condition) required for reaching the correct solution. This finding may be further understood through the analysis of pupils' procedural and conceptual understanding in mathematics. Only those cases where the bar model was used *and* the correct solution reached, demonstrated conceptual understanding. Considering these two findings together, the data suggests that although pupils' procedural understanding enables them to partially construct a bar model, its accurate representation and success in reaching the correct solution, requires conceptual understanding. This finding may have implications for the teaching of the bar model within mathematics lessons (discussed in chapter 6.3). Analysis into the type of bar model used (based on the variation of the model which the question lends itself to), indicates no significant influence of the variation of bar model used on reaching the correct solution, however, may be an area for further research (see chapter 6.3).

To summarise, based on the findings from the current study, the key factors (conditions), which appear to indicate potential significance in reaching the correct solution, when analysed independently (not as configurations of conditions), appear to be:

- Pupils' mathematical attainment;
- Attention and working memory;
- High self-perception of mathematical ability;
- Conceptual understanding (to successfully utilise the bar model);

In terms of the findings from QCA analysis, the study indicates that there is no single condition, which is deemed necessary for reaching the correct solution to the mathematical word problem. However, in terms of sufficiency, absence of the bar model and high mathematical attainment and high levels of self-perception of mathematical ability are all sufficient conditions for reaching the correct solution to the mathematical word problem.

Chapter 5: Discussion

This chapter begins by reminding the reader of the research questions the current study set out to answer. The chapter is structured according to the key findings from the study. Each of the key findings is considered in turn and are analysed against the literature reviewed in chapter 2, using the results from the current study and the existing literature around the topic to support this. Within each section, the limitations of the findings are considered, along with discussing any conflicting findings with the current literature. Throughout the chapter, any unexpected findings from the study are discussed and considered against the existing literature. Whilst the findings are used to partially answer the research questions set out in the study, this is considered in further detail in the conclusion (chapter 6). Finally, the key findings, considered against the existing literature, are used to provide a summary of the main implications of the current study and the recommendations for potential future research, which are elaborated on in the chapter 6.3.

The findings relating to the QCA-specific analyses of the current study are discussed in detail in chapter 5.2, along with a critique of the use of QCA within this study.

As a reminder for the reader, the current study set out to answer the following research questions:

1. **What are the key contextual factors and mechanisms underpinning successful solving of mathematical word problems for autistic pupils?**

2. **Can the exploratory use of qualitative comparative analysis (QCA) be used to determine sufficient and necessary conditions required for autistic pupils solve mathematical word problems?**
    a. Is the bar model sufficient, or does it form a necessary factor within a combination of other conditions, for autistic pupils to solve two-step, real-life mathematical word problems?
    b. Is the bar model sufficient to support autistic pupils in solving mathematical word problems?

c. Does the bar model form a necessary factor within a combination of other conditions to support autistic pupils in solving mathematical word problems?

3. **Is the overall success rate in determining the correct solution to mathematical word problems greater when the bar model is employed by autistic pupils?**
    a. Do autistic pupils, who have been exposed to the bar model, choose this approach when solving mathematical word problems?
    b. Do autistic pupils choose the use of visual representations when solving mathematical word problems?

5.1 Discussion of the main findings

As a reminder for the reader, the initial conditions (for which data was collected), driven by the literature, were as follows (discussed in chapter 3.4.2):

- Reading attainment (RA)
- Mathematical attainment (MA)
- Choose to use the bar model (BM)
- Accuracy of representation of magnitude and relations in visual representations (VROF)
- Self-perception of mathematical ability.

Following preliminary analysis, only those conditions indicating a subset relation of greater than, or equal to, 0.8, were used for QCA analysis (discussed in chapter 4.2):

- Mathematical attainment (MA)
- Not choosing to use the bar model (~BM)
- Self-perception of mathematical ability.

5.1.1 The influence of reading attainment in mathematical problem solving

Within the current study, reading attainment appears to be a redundant condition in mathematical problem solving, as discussed in chapter 4.2. Many studies suggest a positive correlation between pupils' expressive communication or general reading ability and mathematical achievement, particularly amongst the autistic population (Björn et al., 2016; Boonen et al., 2013; Jones et al., 2009; Kintsch & Greeno, 1985; Kleinert et al., 2015; Oswald et al., 2016; Powell et al., 2019; Wei et al., 2015; Whitby & Mancil, 2009). However, in contrast to these studies, findings from the current study align with Ngeno et al.'s (2019) study, who suggests that an individual's ability to read mathematical texts is not indicative of mathematical word problem solving ability (Ngeno et al., 2019). The participants' reading attainment was collected through the teacher interviews in the current study and indicated no correlation to pupils' overall ability to reach the correct solution to the mathematical word problem used in the mathematical task. Both pupil SchAP1 and SchAP2 were working at above ARE for reading but only SchAP1 reached the correct solution to the mathematical word problem. In contrast, SchGP1 and SchBP1 were working below ARE in reading, yet still SchBP1 reached the correct solution. However, it should be considered that in the current study, the mathematical word problems were read to the pupils, thus potentially mitigating the alignment between those findings from Ngeno et al.'s (2019) study. Through the researcher reading the word problems to the participants, any impact of reading fluency may have been eradicated, however, the skills of text comprehension were still required for the pupils to understand the word problem. Through analysis with QCA, reading attainment appeared not to be a significant subset of reaching the correct outcome, thus a redundant condition (discussed further below), consequently aligning with the background case data in the current study. However, to enhance the validity of the current study, future the research would benefit from a wider sample and removal of the researcher reading the word problem to the participants, in order to confidently warrant these claims and to ascertain whether the findings would indicate the same outcome.

Interestingly, the role of EFs, specifically working memory, is suggested to be very similar in both reading comprehension and mathematical problem solving (Björn et al., 2016; Özsoy, 2015; Utami & Warniasih, 2019), which partially aligns with the findings from the current study. Pupil SchGP1 had the greatest number of EF coded items, including working memory (see table 16) and was working below ARE in both reading and mathematics. In contrast to this, SchAP1, who recorded no EF deficits despite being autistic, was working at above ARE in both reading and mathematics, thus aligning with the findings from Björn et al.'s (2016) and Özsoy's (2015) studies. Furthermore, age and gender appeared to be significant in Björn et al.'s (2016) study, which suggests reading comprehension to be more indicative of mathematical word problem solving ability in boys aged 10-11 years, than in subsequent years. As the participants in the current study only consisted of boys, with a mean age of 9 years 11 months (ASD) and 10 years 6 months (NT), further research would benefit from widening the sample to include girls and older children (discussed further in chapter 6.3).

This finding goes some way to answer research questions one and two, in terms of eliminating reading attainment as a key condition for reaching the correct solution to mathematical word problems. The current study suggests that reading attainment is neither a necessary, or sufficient condition, either on its own, or in conjunction with other conditions, to reaching the correct solution to mathematical word problems. Nevertheless, such a claim is made on cautious grounds, as the pupils were not required to read the mathematical word problem in the current study (therefore potentially eliminating any influence of reading fluency), however were required to utilise their text comprehension skills to understand and interpret the problem. This is discussed further below and in more detail in the conclusion (chapter 6).

5.1.2 The influence of mathematical attainment in mathematical problem solving

The findings from the current study indicate pupils' mathematical attainment (which was established from the teacher interviews and based on the schools' current tracking systems) to be a significant condition in reaching the correct solution to mathematical word problems. As a reminder for the reader, pupils' mathematical attainment was determined using the schools' tracking data, indicating the extent to which pupils were working below, at, or above age-related expectations. Subsequently, it must be acknowledged that within this measure of attainment, mathematical problem-solving ability inevitably forms one aspect of this. QCA subset analysis (see chapter 4.2) found mathematical attainment to be a subset of reaching the correct solution (0.897) (see table 21).

Aligning with many previous studies (Aagten-Murphy et al., 2015; Benaron, 2009; Iuculano, 2012; Keen et al., 2015; Wei et al., 2015), the current study found mathematical attainment to be considerably varied amongst the autistic sample. The current study furthermore is supported by previous studies on mathematical problem solving for this population, where the variability of results is substantial (Chiang & Lin, 2007; Keen et al., 2015; Mayes & Calhoun, 2006; Whitby & Mancil, 2009). Within the current study, the mathematical attainment of the autistic subgroup (n=7) ranged from working significantly below ARE (SchAP2, SchGP1) to working at greater depth (SchBP3). The NT cases (n=2) were both working at ARE in mathematics. In both cases where mathematical attainment was significantly below ARE (SchAP2, SchGP1), the correct solution to the mathematical word problem was not reached. In the case of SchBP3, who was working at greater depth in mathematics, along with all other autistic pupils who were working at ARE or above, the correct solution was reached. Only SchBP4, who was working at ARE, did not reach the correct solution – this participant was from the NT group (n=2).

Previous studies have found it difficult to understand the significant variability in mathematical achievement within the autistic population (Aagten-Murphy et al., 2013; Keen et al., 2015). Going some way to answer research question one, the current study may go some way further to explore the mechanisms behind this varied ability in mathematics, by considering other conditions (factors), such as reading attainment (discussed above) and pupils' self-perception of their own mathematical ability (discussed below) and the influence of deficits in specific EF skills. The current study indicates strong evidence for the requirement of mathematical attainment to be at, or above ARE, to successfully reach the correct solution in word problems. However, as discussed above, this measure of mathematical attainment remains a broad category and may encompass pupils' problem-solving ability, hence further study to attempt to separate out these components of mathematical attainment would be beneficial. Furthermore, the current study suggests evidence for the influence of the EFs (discussed in chapter 3.6.1 and in more detail below), particularly working memory and attention, to be potentially significant mechanisms underpinning the varied mathematical achievement for this group.

In terms of QCA analysis (discussed in chapter 4.2), high mathematical attainment (working at ARE or above) may be considered an *inus* condition for reaching the correct solution in the autistic cases. That is, mathematical attainment (of either ARE, or above) is insufficient, on its own, for reaching the correct solution, however, forms a necessary part of the configuration with other conditions, for reaching the correct solution (discussed further below). Despite all autistic cases whose mathematical attainment was at ARE, or above, reaching the correct solution, QCA analysis suggests that this condition, is not a 'necessary' condition – i.e. it is not solely required for reaching the correct solution, but is indeed sufficient in conjunction with other conditions (high self-perception), in reaching the correct solution – hence an 'inus' condition.

5.1.3 The influence of the executive functions (EFs) in mathematical problem solving

Within the current study, the number of EF deficit coded items, assigned to the autistic subgroup ranged from 0-15, with a mean number of 4.7. Of the seven ASD pupils, only one (SchAP1) indicated no EF deficits. In contrast, both NT pupils had no EF deficits coded to them.

The EFs, being a complex set of interrelated processes, are suggested to be significant in the solving of novel problems (Berenguer et al., 2017; Gioia et al., 2000; Goldstein & Naglieri, 2014), in particular aiding the retrieval of learned information (Ullman & Pullman, 2015b), a key requirement in mathematical problem solving.

As discussed earlier, the EFs are suggested to be significant in mathematical problem solving as well as reading comprehension (Björn et al., 2016; Özsoy, 2015; Utami & Warniasih, 2019). In the current study, the three pupils either working below, or significantly below ARE in mathematics (SchAP2, SchGP1, SchCP2) all failed to reach the correct solution and were recorded to have four, or more, EF deficits. Pupil SchGP1 recorded the most EF deficits – 15. These three pupils all recorded deficits in attention and working memory – both EF skills highlighted as being significant in mathematical problem solving (Björn et al., 2016; Özsoy, 2015; Utami & Warniasih, 2019).

Adding further evidence to research question one, working memory and attention are two skills encompassed within the EFs and appear to be potentially significant in the current study. All of those ASD cases within the current study, who failed to reach the correct solution to the mathematical word problem, exhibited impaired EF skills. Moreover, excluding the NT cases, of those cases reaching the incorrect solution, two, or more, coded items relating to attention and/or working memory were recorded (see table 16). Such correlation between working memory deficits and performance on the mathematical problem-solving task goes some way to align with Ngeno et al.'s (2019) suggestion that the relationship between the volume of information presented to an individual to process and the available working memory is indicative of the overall performance on complex tasks.

When considering pupil SchAP1, this case appears to not follow the trend of the other cases, as no EF deficits were recorded – of the seven ASD cases within the current study, six of them displayed some EF deficits. The inclusion of case SchAP1 is an important aspect within the current study, as in order to generate diversity amongst the cases analysed within QCA, cases displaying maximum heterogeneity should be considered (Rihoux & Ragin, 2009) (see chapter 3.2). Such diverse cases may provide evidence to support those findings from the more homogeneous cases, however consideration should be given to the concept of causal asymmetry (chapter 3.2). Although the presence of EF deficits, notably working memory and attention within the current study, appear to be indicative of reaching the incorrect solution, the absence of such deficits (as in SchAP1) and the opposite outcome (reaching the correct solution), should not be assumed to be consistent. In terms of causal asymmetry, any condition (in this case, potentially working memory and attention deficits), which explain the presence of a particular outcome (in this case, reaching the correct solution), may differ from those, which explain its absence. As discussed in chapter 3.2, in terms of set-theoretical methods, the presence or absence (negation) of a condition giving rise to an outcome, or the presence or absence (negation) of an outcome set, refer to two qualitatively different phenomena (Schneider et al., 2013; Thomann & Maggetti, 2017).

It is suggested that, for some autistic individuals, the absence of cognitive deficits (observed at the empirical level as in case SchAP1), may indeed be due to the enactment of an underlying mechanism – compensation (Livingston et al., 2018; Ullman & Pullman, 2015), as discussed earlier. Therefore, within the current study, it must be considered that case SchAP1 may be utilising such compensation strategies to conceal any observable cognitive difficulties (as a result of those EF deficits observed within the other ASD cases), therefore adding to the complex interaction of potentially underlying mechanisms, which may operate at the same time (Danermark et al., 2002), within the problem solving process. It should be remembered here, that mechanisms cannot be considered simply as regularities, but as 'potentially causal generative processes', which, under specific conditions and contexts, can operate (Jones, 2010, p. 203). Furthermore, as in the current study, it is essential that not only the generative mechanisms are explored, but also the conditions in which they operate, therefore giving rise to the observed phenomena within the domain of the empirical (Scott, 2014) and contributing findings to research questions one and two. Consequently, if it is that working memory and/or attention deficits within the ASD subgroup are indeed underlying mechanisms preventing successful solution of mathematical word problems (under specific conditions), a further mechanism of compensation, which may impact upon working memory capacity (Livingston et al., 2018), may interact with the existing mechanisms to produce a different outcome. As this potential mechanism of compensation appears to be significant, this is an area, which would benefit from further future study.

As with many studies, a degree of caution must be applied to the findings from the current study, due to the limited sample size, the heterogeneity of the autistic population, and the influence of wider conditions and mechanisms within such an open system (Danermark et al., 2002), which are not fully considered within the scope of the current study.

A further anomaly in the data can be seen when considering the link between the number of EF deficits coded and pupils' mathematical attainment. Using the findings from the current study, in all ASD cases, except for SchBP3, where the number of coded EF deficits was greater than, or equal to four, the pupils were working at (n=1), or below (n=3) ARE in mathematics. Both cases SchAP1 (with no coded EF deficits) and SchCP1 (with two coded EF deficits) were working above ARE in mathematics. Therefore, the data from the current study indicates a potential link between the number of EF deficits coded, mathematical attainment and reaching the correct solution. The indicative pathways, according to the data from the current study, may be represented as:

Four, or more, coded EF deficits (including working memory and attention) → working at, or below, ARE in mathematics → reaching the incorrect solution to the mathematical word problem.

The only ASD case not fitting this pathway, is SchBP3, who recorded four EF deficits, yet was working at greater depth in mathematics, thus reached the correct solution. This, like SchAP1 discussed above, may be due to compensation strategies mitigating the impact of the EF deficits within this case.

However, these findings may go some way to address the gap in the research into the interactions and combinations of conditions and mechanisms, which may impact upon mathematical problem solving ability within this population (Bae, 2013; Chiang & Lin, 2007; Wallace et al., 2019), as set out in research questions one and two. Nevertheless, to strengthen the current study, the research was carried out within the pupils' school setting, an area of research suggested to be of significant need in this field (Wallace et al., 2019).

In terms of the research questions the current study set out to address, these findings may provide some indication of the potential mechanisms at play, which underpin successful mathematical problem solving (research question 1). Whilst further research would be needed to support these findings, the current study indicates the potential influence of specific EF deficits – namely working memory and attention, as underlying mechanisms within the problem-solving process. However, it must be acknowledged that as in the cases of SchAP1 and SchBP3, further mechanisms, such as compensation, may influence the observed outcome. Furthermore, the current study must also acknowledge the significance of contextual factors and the influence and interaction of other conditions in the potential observable phenomena produced by the mechanisms. The activation of any mechanism may be influenced by the contextual factors and the presence of absence of other conditions in which they reside (Anderson, 2019; Danermark et al., 2002; Scott, 2014). As discussed in chapter 3.1, any claims as to the potential underlying mechanisms (specific EF deficits and compensation within the current study) must be considered within the specific context, and in conjunction with the specific conditions, associated with each case, particularly within open systems as with the current study (Anderson, 2019; Danermark et al., 2002; Harré & Madden, 1975; MacLeavy, 2019; Maxwell, 2012; Pawson & Tilley, 1997). Following on from this, when considering the potential generalisability of any claims, Bassey's (2001) concept of fuzzy generalisations (discussed in chapter 3.1) may be acknowledged. Here, it is important to consider potentially similar outcomes, amongst those 'similar cases' and the through consideration of the context and other conditions present.

5.1.4 The influence of pupils' self-perception of their own mathematical ability in mathematical problem solving

The findings from the current study, align with previous research where pupils' own perceptions of mathematical competence, particularly in problem solving, was indicative of a key component to successful problem solving (McLeod, 1985; Schoenfeld, 1985).

In the current study, where all cases reached the correct solution to the mathematical problem-solving task (ASD and NT cases), pupils' own self-perception of mathematical ability was rated 0.5, or above. A score of 0.5 indicated that the pupil was not unhappy with their own mathematical ability, but it was not necessarily high. Of four pupils, who rated their own self-perception at this level, two of them successfully completed the word problem. In both of these cases, their mathematical attainment was above ARE. All cases (ASD and NT), where self-perception of mathematical ability was rated as high (1), the correct solution was reached. Furthermore, these pupils were all working above ARE in mathematics. The two cases, where self-perception of mathematical ability was rated as low (both ASD cases), the correct solution was not reached. When considering these findings, as discussed in chapter 4.7 above, and further in chapter 6.3 below, consideration must be given as to the potential relationship between these two factors (conditions). Further research into the interplay between these two conditions would be beneficial, to determine whether mathematical attainment is influential of pupils' self-perception of their mathematical ability, or indeed, vice-versa.

The findings from the current research, potentially identify pupils' self-perception of mathematical ability **and** pupils' current level of mathematical attainment both as key conditions (factors) in successful mathematical problem solving. Strengthening this finding, Muis et al. (2015) suggests that the epistemic emotions, both positive and negative, may influence processes, such as self-regulated learning and motivation, which may subsequently impact upon levels of achievement.

Analysis of the findings, using QCA, indicate that both mathematical attainment (discussed above) and pupils' self-perception of their own mathematical ability are subsets (with a consistency score of >0.8) of reaching the correct solution (see chapter 4.6). QCA analysis, based on the data from the current study, suggests that for ASD cases (the NT cases were not included due to the small sample size n=2), a combination of high/or medium levels of self-perception (0.5 or above) ***and*** mathematical attainment of ARE, or above, form a sufficient pathways to reaching the correct solution.

Thus, based on this key finding, coupled with the potential impact of EF deficits, discussed above, a potential causal map, based on the data from the current study, can be constructed as follows, thus far:

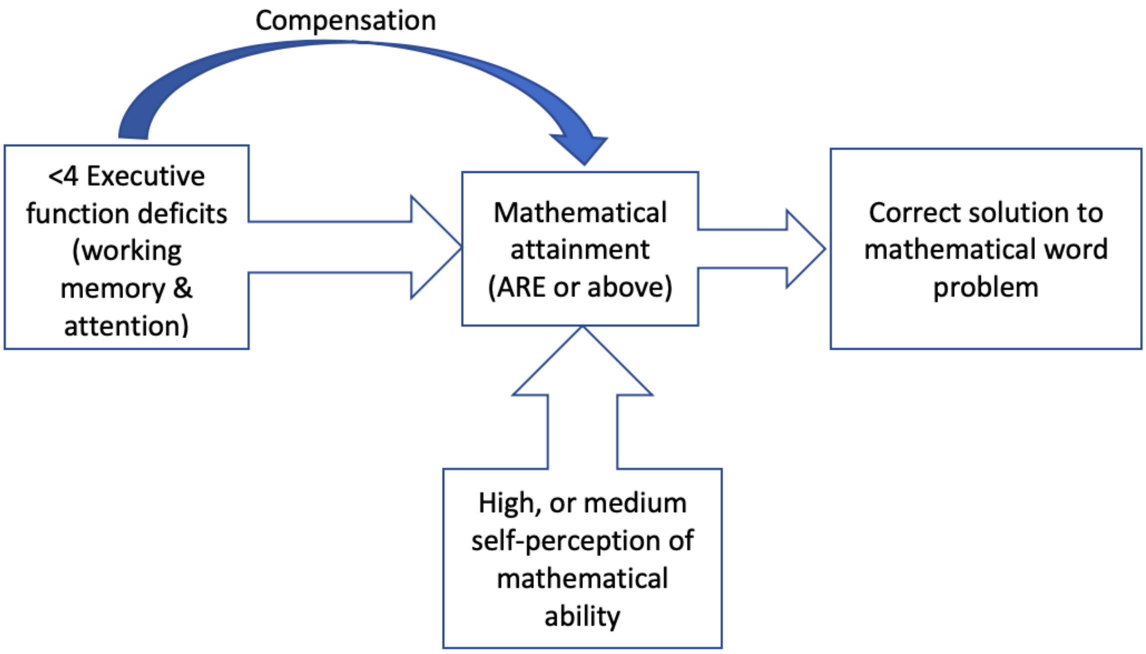


Figure 39: A causal map of conditions and mechanisms leading to the correct solution for autistic cases

5.1.5 The influence of weak central coherence (WCC) on mathematical problem-solving ability

Within the present study, WCC was measured using the stem sentence completion task (Booth & Happé, 2010), which is designed to identify local and global processing tendencies. The sample used within the current study achieved a mean score of 16.4 (ASD) and 18 (NT) (scored out of 20) on the task. The mean number of local processing errors for the ASD group (n=7) was 1.85 (range 0-5), compared to 0.5 (range 0-1) for the NT group (n=2). In the study carried out by Booth & Happé (2010), the mean sentence completion score was 15.63 for the ASD group (n=41) and for the NT group, 17.47 (aged 8-10 years, n=47) and 17.38 (11-13 years, n=40). The mean number of local processing errors for the participants in the study by Booth and Happé (2010), was 1.56 for the ASD group (range 0-5), compared to 0.74 (range 0-6) and 0.90 (range 0-5) for the two NT groups (8-10 years and 11-13 years respectively). The findings from the current study indicate a very similar pattern of results to Booth & Happé's (2010) study, indicating the tendency to produce local processing errors significantly more frequently within the ASD group, compared to the NT group. The findings from the current study align with Booth & Happé's (2010) study, not only with respect to similar mean scores for both subgroups, but also the difference between the mean scores for the two subgroups (1.84 in Booth & Happé's (2010) study and 1.6 in the current study). Furthermore, the mean number of local processing errors found in the current study, align with those in the original study by Booth & Happé's (2010). The slight variation within the findings for each group may be due to the limited sample size of the current study and would therefore benefit from investigation with a larger sample to explore this relationship further. Such similarities between the findings enhance the validity of the stem sentence completion task used within the current study, along with the data produced.

There appears to be no correlation between the number of local processing errors and pupils' mathematical attainment in the current study. As discussed earlier, mathematical attainment appears to be indicative of problem-solving success (and is influenced by the EF skills), whereas WCC does not appear to indicate any potential mechanism within this process.

When comparing the stem sentence completion scores to reaching the correct solution to the mathematical word problem, there appears to be no correlation within the current study. These findings, unlike the potential influence of EF deficits, discussed above, may provide evidence to support Booth et al.'s (2003) suggestion that WCC, although characteristic of autism, is not related to impaired EF skills. The current study indicates that the EF skills may have a significant impact upon mathematical attainment and problem solving, whereas WCC may not.

5.1.6 Use of the bar model, as a visual representation, within mathematical problem solving

Within mathematical problem-solving, the bar model is designed to provide a consistent visual representation to emphasise the magnitude and relationships between known and unknown variables within the problem, thus reducing the demands on cognition and working memory (Maglicco, 2016; Spooner et al., 2017). The findings from the current study have already indicated the potential influence of working memory on mathematical attainment and successful mathematical problem-solving. It is suggested that the bar model may indeed be a successful tool for supporting individuals with learning difficulties (Maglicco, 2016), hence its use within the current study.

Similar to the findings from the study carried out by Bae (2013), where only 3/20 ASD pupils and 4/20 NT pupils selected the use of visual representations during problem-solving, of the nine participants in the present study, only one (ASD), initially chose to utilise this approach. This was despite all pupils being exposed to the bar model for at least two years prior and all questions lending themselves to solution using the bar model, if required by the pupil. Following prompting, six out of the nine participants selected the bar model as an approach to solve the mathematical word problem (five from the ASD subgroup and one from the NT subgroup). The three pupils (2 ASD and 1 NT), who did not use the bar model, or any other visual representation, even after prompting, all went on to reach the correct solution to the word problem. However, it must be acknowledged here that these pupils' mathematical knowledge may have enabled them to reach the correct solution without the requirement of a visual representation to support their understanding. This may be due to a sense of stigma associated with the use of visual representations for those pupils who are less competent in mathematics. Bae (2013) concluded that the use of visual representations is not associated with the word problem-solving abilities of ASD or NT pupils – the findings from the current study appear to align with this conclusion. However, the findings from the current study appear to contrast other studies, which suggest a positive correlation between external schematic representations and problem-solving success (Hegarty & Kozhevnikov, 1999; Peltier & Vannest, 2018; Siregar et al., 2019). Furthermore, in contrast to the findings by Boonen et al. (2013), in which it is suggested that the ability to construct visual schematic representations may be a necessary, but not sufficient condition, within mathematical problem solving, the current study does not fully align with this. Those pupils, who did not use the bar model, but reached the correct solution, did not use any other visual representation to do so, indicative of the potential ability to problem solve for these pupils, without the need for a visual representation. However, when the bar model was used (the only visual representation), the correct solution was reached only when it was constructed accurately. Based on the findings from the current study, QCA analysis did not indicate any configuration where the bar model was necessary, or indeed sufficient, within the problems used. The analyses did, however, indicate that the bar model always requires the presence of other conditions – high mathematical attainment and high levels of self-perception of mathematical ability, to yield the correct solution. As QCA analysis in the current study did not use EF skills as a condition, it may be that the bar model, or other visual representation, forms a necessary part of a larger combination of conditions, such as strong EF skills, to accurately solve mathematical word problems. This is an aspect which would benefit from future research from the current study.

Unlike Maglicco's (2016) findings, where students' problem-solving scores increased significantly following instruction using the bar model, the current study, found no significant benefit from using the bar model. However, in Maglicco's (2016) findings, students were exposed to eight teaching sessions, dedicated to the use of the bar model, whereas the participants within the current study were simply known to have been exposed to the approach during previous maths lessons. This initial lack of choosing the bar model, despite being exposed to it within the mathematics classroom, coupled with the disparity between Maglicco's (2016) findings, may have significant implications for the teaching and learning of this approach (discussed in chapter 6.3). It should, however, be considered that in the current study, no data as to the frequency and consistency of the use of the bar model in the teaching and learning of mathematics was collected – and therefore could not be controlled for. Again, this would be an area for further research within this study, to ascertain the most beneficial approaches to using the bar model within mathematics lessons to maximise pupils' problem-solving success.

Successful utilisation of the bar model requires pupils to first identify the problem type and then represent the magnitude and relations of the numbers in the problem accurately within the schematic representation, before translating this model into a mathematical sentence (Ciobanu, 2015; Maglicco, 2016). Of the six pupils in the current study, who used the bar model (either initially, or after prompting), only two (SchAP1 and SchCP1 - both ASD cases) reached the correct solution to the mathematical word problem. However, it must be acknowledged here that prompting may have influenced the findings, as the pupils may have subsequently used the bar model to satisfy the researcher. Furthermore, of these two, only one (SchAP1) represented the correct magnitude and relations within the representation. However, of the five cases where the bar model was used, prior to, or following prompting, the bar model was not constructed accurately, and the correct solution was not reached. This may align with Hegarty and Mayer's (1995) conclusion that there tends to be more difficulty in constructing the visual representation – the structural phase (Ciobanu, 2015), than performing the required computations to reach the solution. An area for further consideration from these findings is whether the construction of a visual representation, such as the bar model, when solving mathematical word problems adds to the cognitive load experienced by some pupils, rather than reducing it.

Considering the EFs (discussed earlier), such skills are commonly executed within the selection of appropriate strategies, monitoring the effectiveness of the strategy and organising information (Bae, 2013; Goldstein & Naglieri, 2014), all of which are important strategies within the problem solving process (Polya, 1945). When considering this, case SchAP1 was the only participant from the ASD subgroup who indicated no reported EF deficits. Even though this may be as a result of effective compensation strategies (discussed earlier), this may support the findings from Bae (2013) and Goldstein & Naglieri (2014), who suggest the importance of the EF skills within the problem-solving process. Further supporting the link between EF deficits and reaching the correct solution, SchCP1, who utilised the bar model and reached the correct solution, had a low number of coded EF deficits (2). Nevertheless, although reaching the correct solution, SchCP1 did not represent the magnitude and relations accurately within the bar model, thus may be indicative of the influence of the EFs within the structural phase (Ciobanu, 2015) of such a representation, and an area for further research. Thus, it may be those impairments within the EFs do indeed contribute to preventing students selecting the bar model, or successfully working through the stages of application of the bar model, to generate this representation, when faced with mathematical word problems. In all of the ASD cases, where the bar model was eventually applied, but the incorrect solution reached, four, or more, EF deficits were recorded, including working memory, which is one factor suggested to potentially mediate the effects of visual representations (Cooper et al., 2018; Rau, 2017). Therefore, the current research potentially indicates one of the mechanisms associated with successful application of the bar model (and possibly other visual representations) to be linked to the EFs – potentially working memory.

In both cases where the bar model was applied ***and*** the correct solution reached (see figure 40, below), pupils' mathematical attainment was above ARE, thus suggesting this to be another significant condition within successful application of the bar model in the problem-solving process.

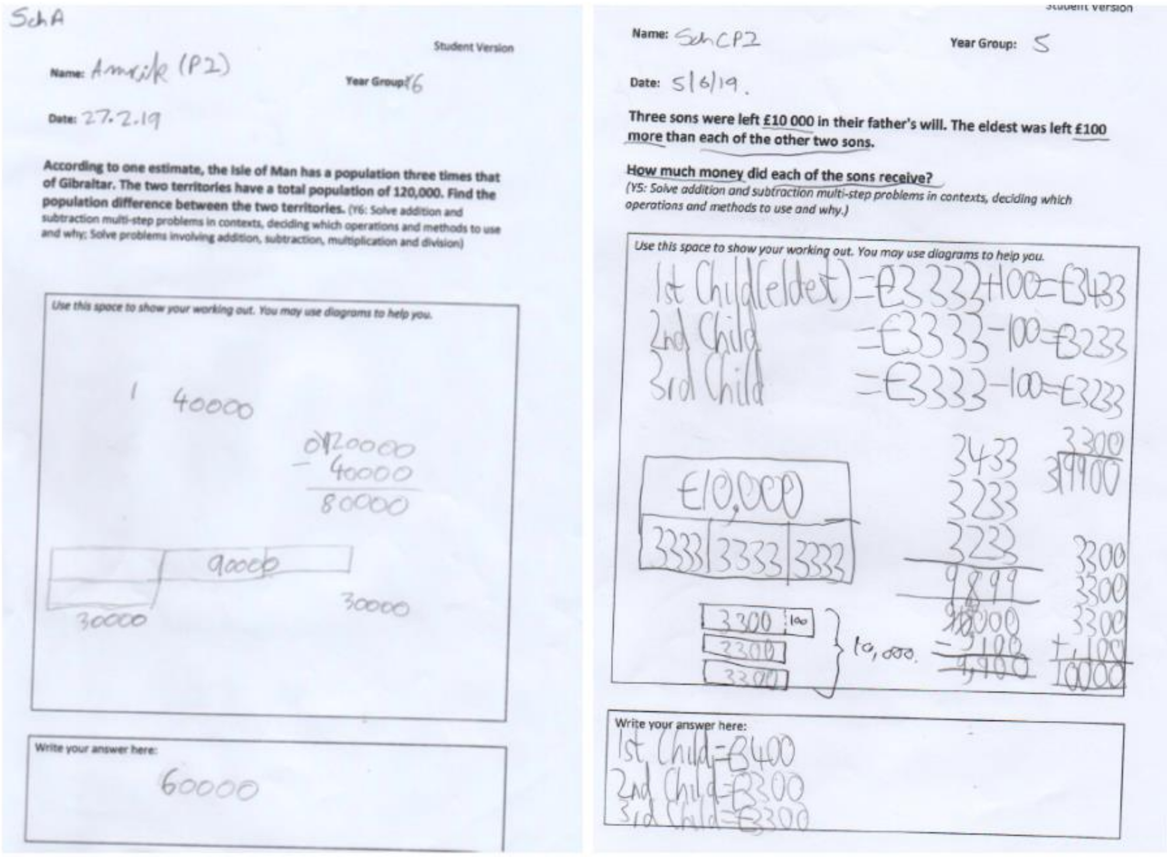


Figure 40: Use of the bar model to solve the mathematical word problems by cases SchAP1 (left) and SchCP1 (right)

In both cases exemplified in figure 40 above, the bar model was used to represent the structure within the word problem, however, additional mathematical knowledge was then utilised to interpret the representation, and correctly solve the word problem, aligning with higher levels of mathematical attainment.

When considering case SchAP2, who, after prompting, chose to use the bar model, yet reached the incorrect solution, it was clear from the interview transcript, as seen in figure 41 below, that he was significantly distracted by information relating to days of the week presented within the question.

R: Four. So, there's a little bit of a trick here. Because actually what they did is they laid 106m every day, for four days…and then, on Friday, they only laid 100m. So, there was a little bit of a trick in that question. Shall I show you how we might do it?

P2: Sure.

R: We could do it your way – with the column method, or the bar model.

(R models the calculation to P2)

P2: They say on the closing and opening times on boards where for shops and all the shopping centres and stuff. They say like, Mon, Tue, Wed, Thurs, Fri…

R: Mhm

P2: Sat, Sun.

R: They do, they do it shorter don't they? So, they don't write the whole word. I've done it even shorter than that. So, on Monday it was 106m, Tuesday 106m, Wednesday 106m, Thursday 106m and how much on Friday?

P2: 100.

Figure 41: Part of the interview transcript from case SchAP2, indicating distraction, based on information within the word problem

As the problem solving process requires pupils to filter information and selectively ignore distractions, whilst following a process (Goldstein & Naglieri, 2014; Keenan et al., 2019), this particular case may indeed highlight the influence of impaired EF skills, including working memory, on the problem solving process (Desaunay et al., 2019; Utami & Warniasih, 2019).

5.1.7 The significance of procedural and conceptual understanding within the application of the bar model

The findings from the current study indicated the significance of conceptual understanding for reaching the correct solution to mathematical word problems. Amongst the autistic cases, where the bar model was used, and the correct solution reached (SchAP1), conceptual understanding in mathematics was at its greatest (see figure 36). In contrast, where the bar model was used and the incorrect solution reached (n=4), all cases demonstrated procedural understanding only, with no coded items for conceptual understanding. Supporting the research by Williams et al. (2015), these findings suggest that difficulties in the flexibility of abstract reasoning within the autistic population may lead to widespread difficulties with conceptual organisation. The only anomaly to these findings was case SchCP1 (ASD), where the bar model was used, the incorrect solution reached, but some conceptual understanding (15% of coded items) demonstrated. This case, although an anomaly, may be significant, as the magnitude and relations were not represented accurately in the bar model, and consequently the incorrect solution reached. The use of atypical cases can provide significant information, as in the current study, to support the overall trends (Stake, 1994). Aligning with the findings from Mutawah et al. (2019), this case may support the evidence for the requirement of secure conceptual understanding for the accurate representation of magnitudes within the bar model and subsequent correct solution to the word problem. In support of this, when considering the structural phase of Ciobanu's (2015) suggested phases of application of the bar model, a secure understanding of the part-whole relationships (on which the bar model is based), is required to successfully execute this phase. When analysing the response provided to the mathematical problem solving task by cases SchAP2 and SchCP2 (see figure 42, below), it is evident that this lack of understanding of the part-whole concept is present, therefore disrupting the successful utilisation of the bar model at phase 2 (Ciobanu, 2015).

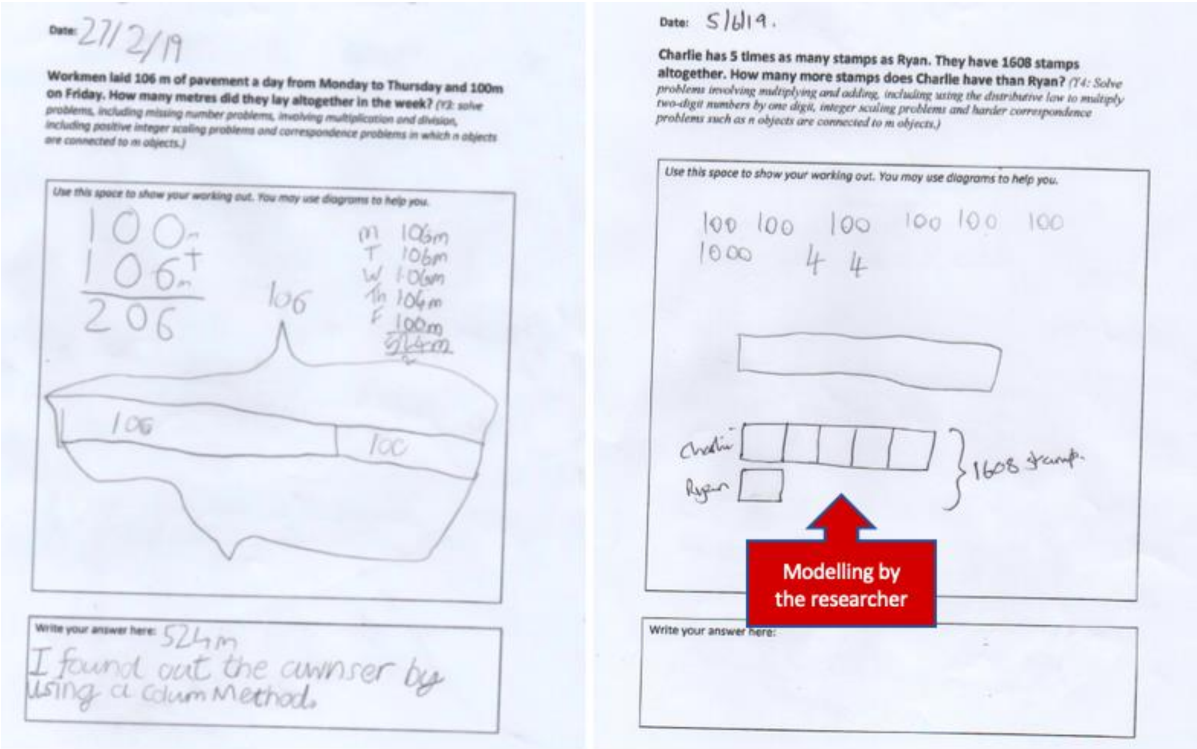


Figure 42: The attempts at using the bar model to solve the word problem, however, demonstrating a lack of conceptual understanding of the part-part-whole relationships for cases SchAP2 (left) and SchCP2 (right)

However, Morin et al. (2017) identify significant debate over whether the bar model provides procedural, or conceptual, support for pupils in the problem-solving process. The findings of the current study indicate the requirement for conceptual understanding for accurately using the bar model to reach the correct solution, however, also support the need for a level of procedural understanding (see below).

When considering the response to the mathematical word problem task completed by case SchAP2 (see figure 42, above), in conjunction with the interview transcript and the excerpt in figure 43 below, it is evident that the pupil had procedural understanding, in terms of how to construct a bar model. Furthermore, the approach taken by SchAP2 aligns with previous research suggesting that a caveat of visual representations is that some pupils place so much focus on the construction of the representation, that the mathematical understanding is lost (Dufour-Janvier et al., 1987).

(Pupil begins drawing bar model)

P2: Ok, is that how long?

R: Yeah, it could be.

P2: And 100 is…that long?

R: Mhm. Ok.

P2: Should I write 106 in here?

R: (acknowledges)

P2: And…so…so we don't need to do this…squidgy lines.

R: Ok.

P2: Hmmm. It is 106 and 100 metres…I think I write 106 on top?

R: Ok.

P2: So, what goes at the bottom squidgy line? I don't know?

Figure 43: Excerpt from interview transcript with case SchAP2, indicating some procedural understanding with relation to construction of the bar model

This pupil was therefore able to apply 'previously identified classification rules' to carry out the required action sequence to solve the problem (using the bar model), however, had significant difficulty in accurate representation (and therefore understanding) of the concepts within the problem (Rittle-Johnson et al., 2001, p. 346; Williams et al., 2015, p. 861). The process followed by this pupil mirrors Skemp's (1971) phrase, 'rules without reasons' (p.12).

Furthermore, the link between the number of EF deficits coded, and the proportion of conceptual understanding items coded, appears to be significant. Those cases where no EF deficits were recorded (SchAP1 (ASD), SchBP2 (NT) and SchBP4 (NT)) indicated significant conceptual understanding (100%, 35% and 100% items coded respectively). With the exception of SchCP1 (ASD), who recorded 15% conceptual understanding and two recorded

EF deficits, the other four cases recorded four, or more, EF deficits and 65%, or greater, procedural understanding. Three of these cases, reported 100% procedural understanding; these findings suggesting deficits in the EFs to potentially impact on pupils' ability to understand mathematical concepts clearly. These findings may be significant in terms of explaining the significance of the EFs in conceptual understanding and highlight the importance of the EF skills in the steps of planning and executing within the problem solving process (Polya, 1945; Roelofs et al., 2015).

Through considering the key findings of the current study, discussed within this chapter, figure 44, below, presents the reader with an overview of the conditions and mechanisms, which appear to be apparent in the problem-solving process for autistic pupils.

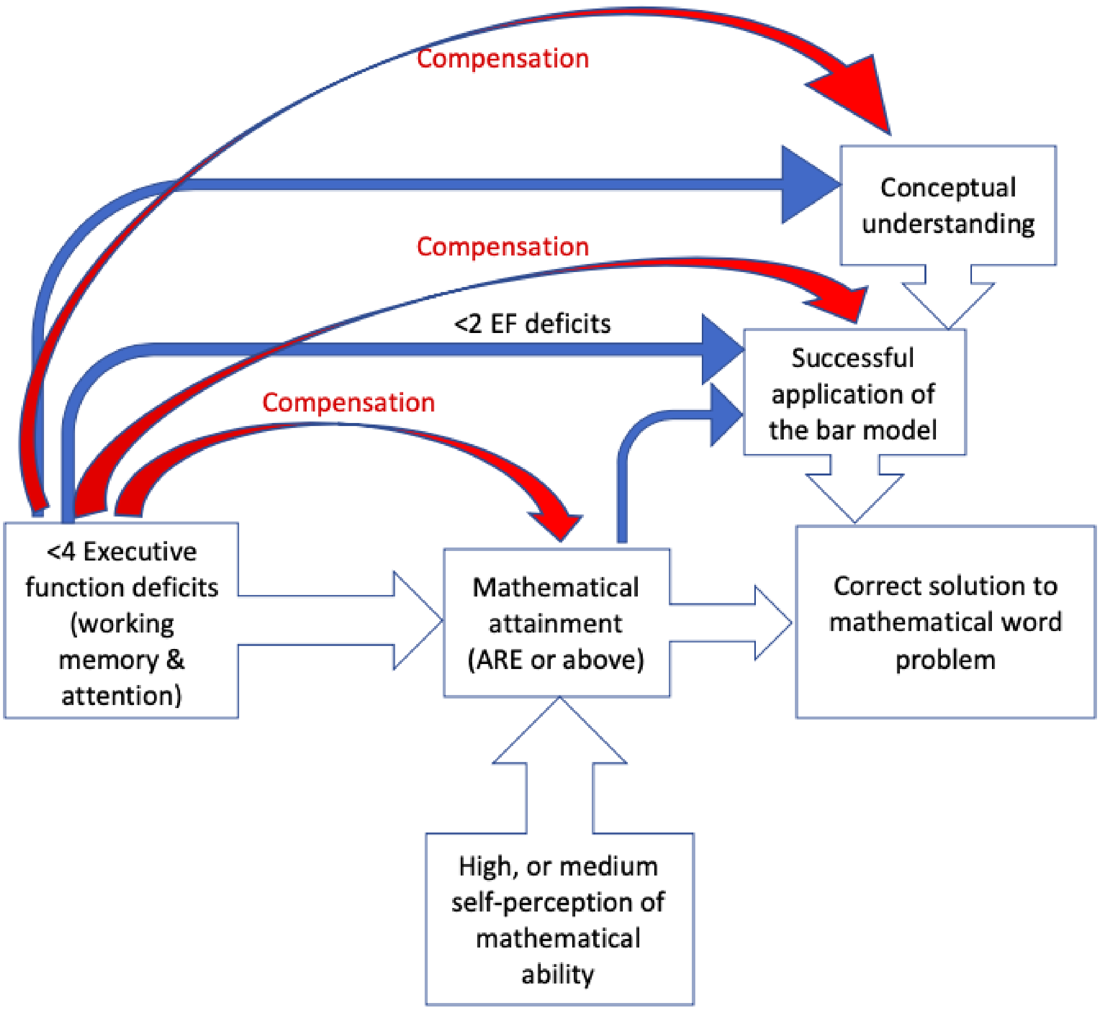


Figure 44: The potential conditions and mechanisms involved in mathematical word problem solving and application of the bar model

Figure 44, above, shows the potential mechanistic pathways and interrelation of the underlying mechanisms, which appear to be apparent within mathematical word problem solving and the application of the bar model. The red arrows indicate where the potential mechanism of compensation (i.e. the masking of potential EF deficits) may influence these processes. The blue arrows represent the interaction between, and the pathways for, different mechanisms. For example, as indicated, where ≤4 EF deficits were recorded, conceptual understanding was observed. In turn, this conceptual understanding supported the correct use and representation of the bar model, which, in turn, gave rise to the correct solution to the mathematical word problem.

An alternative pathway can be seen where there is evidence of ≤2 EF deficits, which appeared to mitigate the emphasis on conceptual understanding for successful use and representation of the bar model. Consequently, figure 44 enables the probable mechanisms (such as EF skills, conceptual understanding, and compensation) to be considered alongside the specific conditions used for analysis within the current study.

5.2 Discussion of the findings within QCA analysis and a critique of QCA, as a methodology, in small-N educational research

This chapter provides a critique of the use of QCA in small-N educational research – one of the contributions to knowledge in the current thesis. Through discussing the findings from the data analysis using fsQCA 3.0 (Ragin & Davey, 2016), along with those findings from manual configurational pathway analysis and the other key findings from the study (discussed in chapter 4), the application of QCA within the current study is reviewed in order to provide evidence towards research question 2.

Aligning with critical realism, QCA sets out to explore the multiple complex configurations of conditions necessary for a specific outcome (or not) (Anderson, 2019). The aim of the current study was to attempt to establish any necessary, or sufficient, conditions, or configurations of conditions, required for autistic pupils to correctly solve mathematical word problems, with a focus on the use of the bar model as a visual representation to support mathematical problem solving. Consequently, the use of QCA, as a methodological framework, was applied to the study in order to understand the conditions and mechanisms underlying successful mathematical word problem solving within this population (Anderson, 2019; Harré & Madden, 1975; MacLeavy, 2019). Although most commonly applied to large-scale studies, often in the field of political science, public policy and comparative sociology (Rihoux, 2013; Roig-Tierno et al., 2017), QCA is well-suited to small-N studies, as with the current study, where N is too large for in-depth case study, but too small for large scale statistical analyses (Jopke & Gerrits, 2019; Thiem, 2014). Furthermore, QCA is best suited to purposively selected samples and explorative research designs, as in the current study (Rihoux & Ragin, 2009; Thomann & Maggetti, 2017). Additionally, as intended within the current study, QCA is well-suited for constructing empirically founded theories, emphasising causal complexity, which seek to identify those causal pathways giving rise to an outcome of interest (reaching the correct solution in the current study) (Hirzalla, 2020; Rihoux & Ragin, 2009; Thiem, 2018).

Based on the arguments discussed above, the current study set out to apply QCA, as a methodological framework, to a small-N educational study, to explore its usefulness within the field.

Five initial conditions were used within the current study, which were derived from the literature (pupils' reading attainment; pupils' mathematical attainment; use of the bar model; measure of correct representation of magnitude and relations within visual representations; and pupils' self-perception of their own mathematical ability). Following subset analysis (see chapter 4.6), this was reduced to three conditions (pupils' mathematical attainment; use of the bar model; pupils' self-perception of mathematical ability). Although these three conditions were used within the final data analysis using fsQCA 3.0 (Ragin & Davey, 2016), and based on a consistency score of >0.8 (see chapter 3.6.2 and 4.6), this potentially identified one of the limitations of QCA within small-N research. The use of small-N within the study limits the quantity of empirical data available (and provided justification for the higher consistency threshold of 0.8 to be used, as discussed in chapter 4.10). Subsequently, where there is insufficient empirical data to support the number of logical possible combinations of conditions, where the cases and reality are not fully diversified in terms of the conditions - the issue of limited diversity is reached (Hirzalla, 2020; Schneider & Wagemann, 2010). The consequence of limited diversity results in logical remainders within the truth table – instances where there are no cases to support the particular configuration. The final truth table (table 29 in chapter 4.6), clearly exemplifies the issue of limited diversity, where, from a possible eight configurations of conditions (2^n, where n represents the number of conditions – 3 in the current study), six configurations represent logical remainders. Consequently, the analysis indicated that there was insufficient empirical data from the number of cases used, to represent all the possible combinations of conditions analysed. Thus, in the current study, where N=7 (the autistic cases used within the analysis), the issue of logical remainders is apparent. Furthermore, as discussed in chapter 4.6, separate QCA analysis on the data from the NT subgroup (n=2), was not feasible within the current study, as the number of conditions (3) exceeded the number of cases, thus a significant lack of sufficient empirical data to support the number of possible configurations. As the aim of the current study was to consider the conditions and mechanisms associated with autistic pupils, the small-N of the NT sub-group was used to provide a limited level of comparative data. Consequently, the small-N used and the number of logical remainders within the current study, the consistency threshold of 0.8 was selected by the researcher. Nevertheless, further analysis using slightly different consistency thresholds may have allowed the researcher to explore the sensitivity of the results within the current study (Schneider & Wagemann, 2013). Also, further research, drawing on a larger sample of NT pupils (as discussed below), would be beneficial to establish a deeper understanding of whether these conditions and mechanisms are likely to be significant and influential in mathematical word problem solving amongst the NT population. A larger-N would also support the process of adjusting the consistency thresholds to explore more fully the sensitivity of the data.

Consequently, due to the limitations discussed above, a comparison of the sufficient, or necessary, configurations, or conditions, between the autistic subgroup and the NT subgroup was not possible. Furthermore, due to the small N used within QCA analysis (N=7), although the current study only analysed three final conditions, the number of configurations still exceeds the number of cases, increasing the likelihood of limited diversity, as seen in the results.

Despite the conditions for analysis within the current study being driven by the existing literature, it is important to consider the significance of the influence of structures and mechanisms operating within the open systems in which the cases were bound (Anderson, 2019; Danermark et al., 2002; Fleetwood, 2017). Consequently, in small-N research, where the number of conditions to be analysed must be kept to a minimum, the risk of additional mechanisms, which are not analysed, operating within such open systems, and consequently influencing the outcome, must be acknowledged. As such, mechanisms such as the social norms present within classrooms must be acknowledged as having potentially causal powers, which the current study does not allow for. Furthermore, conditions such as the frequency and pedagogical approach used when teaching the bar model, along with the influence of the type of bar model used (comparison or part-part-whole – discussed in chapter 4.5) must also be acknowledged as potential conditions influencing the outcome. Nevertheless, due to the small-N used within the current study, the limitation on the number of conditions for analysis, undoubtedly resulted in some possible significant conditions (which may indeed prove to be necessary, or sufficient) to be excluded from the study. However, in defence of this argument, Rihoux and Ragin (2016) acknowledge the necessity of any generalisations from QCA to be 'modest' (p.11) and to acknowledge that no claim is made that those conditions not included in the analysis (for example those mentioned above) would not affect the results of the study.

However, when analysing the results from QCA analysis, using fsQCA 3.0 (Ragin & Davey, 2016), the findings align with the separate analysis of case data. Aligning with the findings discussed in chapter 4.6, QCA analysis in the current study, indicated the significance mathematical attainment and pupils' self-perception of their own mathematical ability, in reaching the correct solution to the mathematical word problem.

Figure 45 (below) indicates the overlap between the findings from QCA analysis of the data within the current study and the findings from individual case analysis. Consequently, this overlap in findings from the current study goes some way to demonstrate the use of QCA analysis to enable a deeper and more nuanced understanding of factors sitting behind successful mathematical word problem solving amongst autistic pupils.

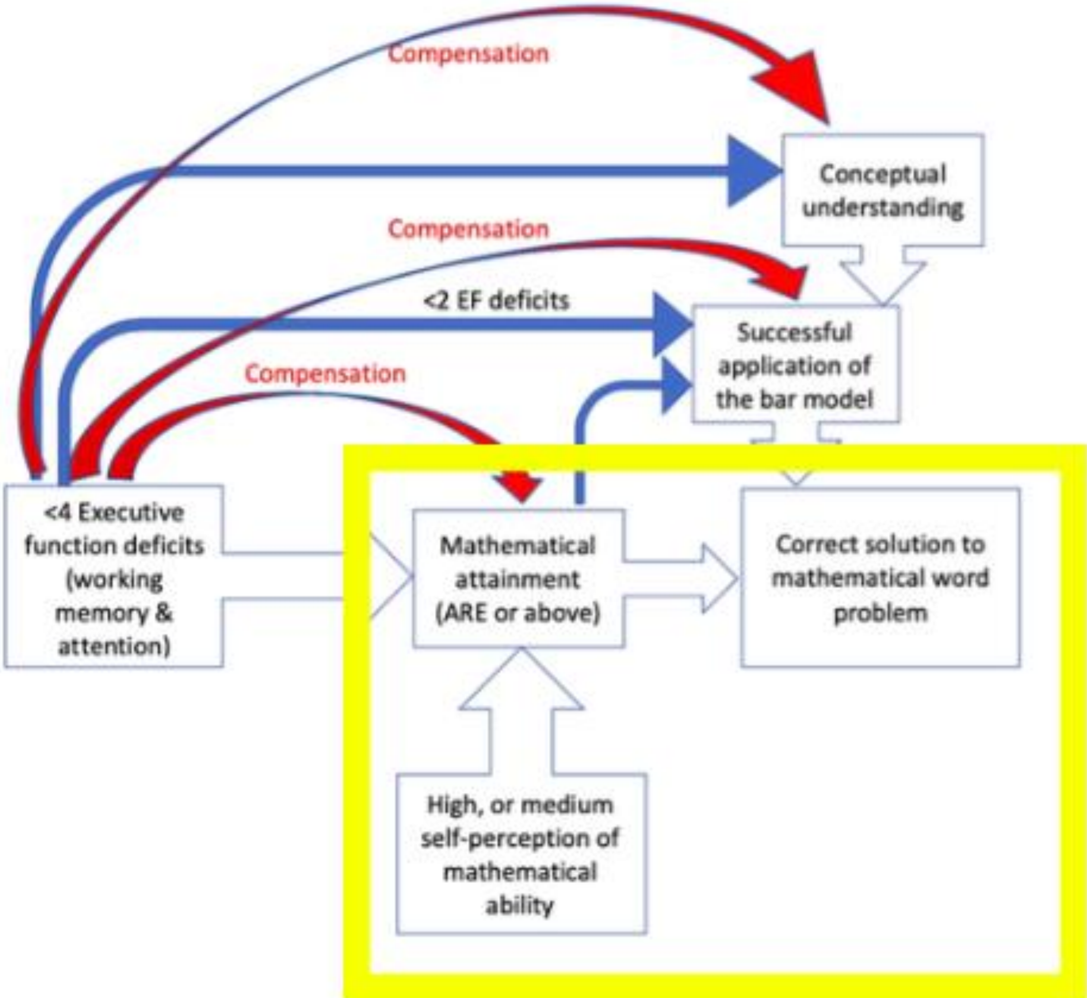


Figure 45: The overlap (indicated by the yellow box) of the findings from the QCA analysis and the case-based analysis within the current study

The minimum formula for sufficiency, derived from QCA 3.0 (Ragin & Davey, 2016), as discussed in Chapter 4.6, was ~BM+*MAtt*SPerc →CorrSoln. Aligning with the findings discussed throughout chapter 4, both mathematical attainment and pupils' self-perception of mathematical ability appear to be key conditions for autistic pupils to reach the correct solution to mathematical word problems, within the current study.

Furthermore, QCA analysis indicates that pupils' reading attainment is a redundant condition, aligning with the findings discussed in chapter 4. QCA analysis indicates that reading attainment is not a subset of the outcome (reaching the correct solution) and its presence, or absence (negated) has the same outcome (see chapter 4.6). The occurrence of reading attainment as a redundant condition, is further supported through the manual analysis of configurations (discussed in chapter 4.6).

Such manual analysis, which although does not allow for the consideration of configurations where there is no empirical evidence available (logical remainders), indicates the significance of both mathematical attainment and pupils' self-perception of mathematical ability to be significant conditions in reaching the correct solution. All analyses (fsQCA 3.0, manual analysis and case analysis) indicate the need for mathematical attainment and self-perception to be at least at the crossover point (0.5) or greater, for the correct solution to be reached.

Despite the acknowledged limitations of QCA, especially within small-N research, it is suggested that any findings may be cautiously generalised to cases, which are similar to those within the study, in terms of background characteristics and that replication in subsequent studies should be possible (Hirzalla, 2020; Roig-Tierno et al., 2017). In addition, those cases, beyond the study, which share the same configurations of conditions to those within the study, may support the generalisations made from the findings (Rihoux & Ragin, 2009).

5.3 Summary

This chapter has attempted to provide a detailed discussion of the key findings from the current study, reviewing these against the current body of literature. Aligning with many previous studies (Aagten-Murphy et al., 2015; Benaron, 2009; Iuculano, 2012; Keen et al., 2015; Wei et al., 2015), mathematical performance and attainment of the autistic population is varied and has a significant influence on overall problem solving ability. Coupled with this key condition, pupils' self-perception of their own mathematical ability has been highlighted as a key condition within the problem solving process, aligning with other studies in this area (McLeod, 1985; Schoenfeld, 1985).

In support of research question one and two, underpinning these two conditions, appears to be several mechanisms, specifically influencing mathematical attainment. In line with previous studies (Berenguer et al., 2017; Desaunay et al., 2019; Gioia et al., 2000; Goldstein & Naglieri, 2014; Keenan et al., 2019; Spooner et al., 2017), the EFs, particularly working memory and attention, are seen as essential to high levels of mathematical attainment in the current study. However, the current study has also indicated the potential mechanism of compensation strategies for masking any EF deficits, as in Livingston et al.'s (2018) study.

In addition to influence of the EFs, as with Mutawah et al.'s (2019) findings, the current study indicates the successful application of the bar model, as an approach to reaching the correct solution to mathematical word problems, appears to be reliant upon pupils' having a conceptual understanding of the problem.

Therefore, based on the findings from the current study, it can be argued that the bar model is not a sufficient condition, on its own, to reach the correct solution for autistic cases. For autistic pupils, high self-perception, high mathematical attainment, and/or high reading attainment, are sufficient for reaching the correct solution, when combined with the use of the bar model, but not necessary conditions.

5.4 Refinement of the conceptual framework

Through drawing on the data analysis and findings discussed in the current study, and figure 45 above, we can return to the conceptual framework, discussed in chapter 2.3. Figure 46, below, indicates the refinement of the conceptual framework developed in chapter 2.3, based on the findings from the current study.

Within figure 46, the highlighted factors and red arrows indicate those conditions and mechanisms deemed significant within mathematical problem solving for autistic pupils, based on the current study. The dotted red lines represent the potential for the mechanism of compensation to be influential within this process. To add further detail to the original conceptual framework underpinning the current study, the findings have enabled the addition of some quantitative data pertaining to the number of EF deficits, which appear to be significant within the mathematical problem-solving process. Therefore, the current study indicates that it is not merely EF deficits, which may influence the problem-solving process, but more specifically, a cut-off of four deficits (or 2 when considering application of the bar model).

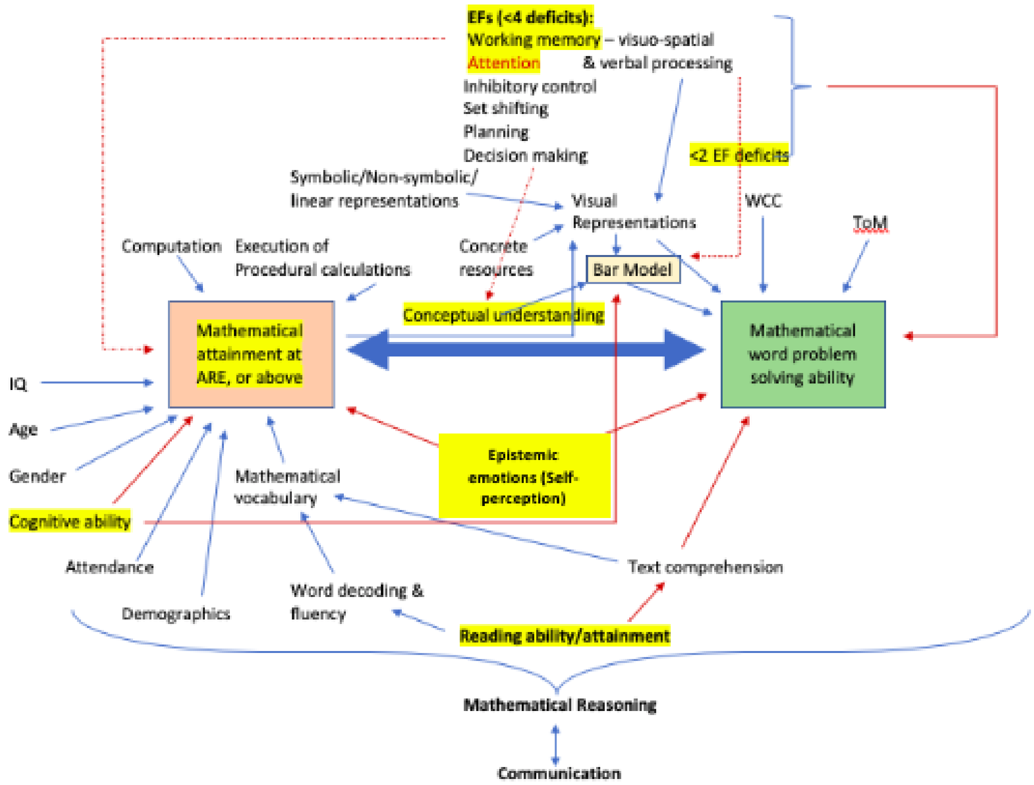


Figure 46: Refinement of the conceptual framework, based on the findings from the current study

Whilst it may be the case that other EF skills play a significant role within mathematical problem solving, the findings from the current study indicated the emphasis on attention and working memory. Those conditions highlighted yellow in figure 46, above, indicate those to be significant within mathematical problem solving amongst autistic pupils, based on the findings from the current study. The potential inter-relationship between mathematical attainment, self-perception of pupils' own mathematical ability and successful mathematical problem solving, can be seen within the refined conceptual framework. As discussed earlier, and further in chapter 6, this inter-relationship has been highlighted as an area for further study. Although other conditions and mechanisms within the conceptual framework may well be significant in mathematical problem solving for this population, as suggested by previous empirical studies and literature within the field (discussed in chapter 2), the findings from the current study either found no indication of these, or did not include these within the conditions for analysis within the study.

The current study has added to the depth of understanding of mathematical problem solving for autistic pupils, through building upon findings from previous empirical studies and contributing to the development of new knowledge within this field, as discussed in detail in chapter 6, below. The refinement of the conceptual framework (figure 46, above) has indicated the alignment between the findings from the current study with the existing literature, with regards to the conditions of mathematical attainment, epistemic emotions (self-perception) and, to a certain extent, the influence (and complexity) of reading comprehension. Furthermore, the study has added to the current body of empirical research through developing a deeper understanding of the role of the EFs, particularly attention and working memory, within mathematical problem solving. The specific focus on the use of the bar model has enabled the findings to provide a further understanding of the mechanisms and factors – conceptual understanding and the role of the EFs, within the successful utilisation of this representation within mathematical problem solving. Such findings may support classroom practitioners to develop a further understanding and to drive pedagogical approaches to support autistic pupils within the mathematics classroom, as discussed further in Chapter 6 below.

Chapter 6: Conclusion

As discussed in chapter 1, one of the main challenges facing researchers in the area of autism, is governed by the heterogeneity of the condition, coupled with the uneven profiles of abilities across different domains, including mathematics (Aagten-Murphy et al., 2013; Agrawal, 2013; Chiang & Lin, 2007; Wallace et al., 2019; Wen, 2018; Whitby & Mancil, 2009). In addition, currently, any specific factors, or sub-group, directly linked with mathematical ability amongst this population, remain unclear (Keen et al., 2015; Powell et al., 2019; Wei et al., 2015). A search of the literature indicated that, as yet, the use of the bar model, as a tool for supporting problem solving abilities within the autistic population, has not been explored. Finally, in contrast to the effectiveness of short-term intervention, little research, particularly which encompasses qualitative elements, has been carried out into the field of more general teaching approaches and strategies to support the mathematical understanding of autistic pupils. Consequently, the current study sought to address these gaps in the literature through the exploration of mathematical problem solving amongst autistic pupils, with a specific focus on the bar model, as one type of visual representation within mathematics. The aim of the study was to identify the key contextual factors and mechanisms underpinning mathematical word problem solving for this group of pupils. As the bar model is one of the visual representations frequently used within the teaching and learning of mathematics, the study sought to understand whether the success rate in determining the correct solution to mathematical word problems amongst autistic pupils, was enhanced when this approach was utilised. Furthermore, the use of the bar model, was a central focus of the current study, to determine which (if any) conditions, or combinations of conditions, were deemed necessary, or sufficient[19], in order to reach the correct solution to mathematical word problems, using this approach. Finally, as the study was interested in determining combinations of conditions associated with mathematical problem solving, the methodological framework of QCA was applied. However, due to its novelty within small-N, educational research, the methodological approach was also used in an explorative manner, in order to review its effectiveness within such research.

[19] Note the use of the terms ‘necessary’ and ‘sufficient’ here. These terms are used as concepts within QCA and the context of the current study, rather than indicating generic concepts of necessity or sufficiency.

The current chapter begins by providing responses to the research questions set out in the study, through drawing on the findings and adding to the current body of literature in the field. A summary of the key contribution to knowledge made through this study is then presented to the reader. Following on from this, the limitations of the current study are acknowledged and discussed. Finally, the implications for both classroom practice, and future research arising from the current study, are discussed, to consider how the current study serves as a driver for future research in this field.

6.1 Responses to the research questions

Drawing on the findings from the current study, a response to each of the research questions, is provided. The responses are discussed against the current body of literature, as well as considering the contribution to knowledge in this field.

Research question 1: What are the key contextual factors and mechanisms underpinning successful solving of mathematical word problems for autistic pupils?

Aligning with previous studies, the current study found mathematical attainment amongst the autistic population to be varied, and to have implications on overall problem solving ability (Aagten-Murphy et al., 2015; Benaron, 2009; Iuculano, 2012; Keen et al., 2015; Wei et al., 2015). However, unlike mathematical attainment, the current study suggests that reading ability is less influential. The current study suggests that reading attainment is neither a necessary, or sufficient condition, either on its own, or in conjunction with other conditions, to reaching the correct solution to mathematical word problems. Nevertheless, although partially aligning with Ngeno (2019), who suggests that an individual's ability to read mathematical texts is not indicative of mathematical word problem solving ability (Ngeno et al., 2019), the current study placed a greater emphasis on text comprehension, rather than reading fluency. Despite the indicative influence of mathematical attainment, and lack of influence of reading ability, the current study proposes that the role of the EFs may be mechanisms underpinning the influence of these conditions. Whilst it has been previously suggested that the EFs are significant in both reading comprehension and mathematical problem solving (Björn et al., 2016; Özsoy, 2015; Utami & Warniasih, 2019), the current study indicates this influence to be more profound (specifically with respect to attention and working memory deficits) within the domain of mathematical problem solving (and mathematical ability in general) for this population. Nevertheless, aligning with Livingston et al.'s (2018) findings, the current study also indicated the potential strategy of compensation for some individuals within this population, as a way of 'masking' EF deficits. Specifically, the current study suggests that those autistic pupils indicating four, or more, EF deficits (not accounting for compensation), are likely to be working at, or below, ARE in mathematics, and consequently, are more likely to reach the incorrect solution to mathematical word problems.

Aligning with previous research, the current study found that epistemic emotions, in this case pupils' self-perception of their own mathematical ability, to be a significant condition within successful word problem solving in mathematics (McLeod, 1985; Muis et al., 2015; Schoenfeld, 1985). However, what is not clear at this stage, is any potential link or relationship between this condition and that of mathematical attainment.

Whilst the current study indicated the significance of conceptual understanding for reaching the correct solution to mathematical word problems for autistic pupils, supporting the research by Williams et al. (2015), once again, the underlying influence of the EFs appeared to be significant here. The current study highlighted the potential mismatch between placing considerable emphasis on following rules and procedures, an aspect of procedural understanding, and the ability to direct attention and focus on understanding the mathematical concepts within the problem, hence reaching the correct solution.

Therefore, in summary, the current study indicates the following conditions and mechanisms to be significant in successful mathematical word problem solving amongst autistic pupils:

- Mathematical attainment (at ARE, or above);
- High levels of self-perception of pupils' own mathematical ability;
- Secure levels of conceptual understanding within mathematics;

- Less than, or equal to, four identified EF deficits, specifically with respect to attention and working memory (unless the strategy of compensation is applied).

Therefore, to provide a response to research question 1, the current study suggests that for autistic pupils, high self-perception and high mathematical attainment are sufficient for reaching the correct solution (when combined with the use of the bar model), but not necessary conditions.

In terms of *inus* conditions, discussed in the chapter 3, high self-perception and high mathematical attainment, can all be considered as *inus* conditions with and without the bar model, for autistic cases. That is, they are each insufficient, on their own, for reaching the correct solution, however they are a necessary part of the configuration with other conditions, including the bar model, for reaching the correct solution.

Nevertheless, and perhaps of greater significance, the explanation for these findings can be related to the underlying mechanisms of the EFs – specifically attention and working memory. However, as Thiem (2018, p.3) points out, 'just because some factor does not appear in the final model, does not mean that this is causally irrelevant to the analysed outcome; it simply means that, conditional on the available data, there exists no evidence for the causal relevance of that factor.' Therefore, the current study makes no claim that additional conditions, not included within this study, will not also influence successful mathematical problem solving for this population.

Research question 2: Can the exploratory use of qualitative comparative analysis (QCA) be used to determine sufficient and necessary conditions required for autistic pupils solve mathematical word problems? Is the bar model sufficient, or does it form a necessary factor within a combination of other conditions, for autistic pupils to solve two-step, real-life mathematical word problems? Is the bar model sufficient to support autistic pupils in solving mathematical word problems? Does the bar model form a necessary factor, within a combination of other conditions, to support autistic pupils in solving mathematical word problems?

The findings from the current study run counter to those proposed by Boonen et al. (2013), who suggests that pupils' ability to construct visual schematic representations may be a necessary, but not sufficient condition, within mathematical problem solving. Within the current study, the use of the bar model (or any other visual representation) was not deemed to be a necessary condition for reaching the correct solution to the mathematical word problem. However, when the bar model was used, its application was not indicated to be sufficient for reaching the correct solution, as a condition of its own. Although not necessary, its use formed part of a sufficient pathway, when in conjunction with other conditions: mathematical attainment at ARE, or above; and high levels of pupils' self-perception of mathematical ability. Although reading attainment at ARE, or above, was a condition within some sufficient pathways, its absence from others suggested it to be a redundant condition.

The findings from the current study suggest that the bar model is not a sufficient condition, on its own, to reach the correct solution for autistic cases. However, high self-perception, high mathematical attainment, and/or high reading attainment, are sufficient for reaching the correct solution, when combined with the use of the bar model, but not necessary conditions. Nevertheless, in terms of QCA, high self-perception, high mathematical attainment and high reading attainment[20], can all be considered as *inus* conditions within mathematical word problem solving, both with and without the use bar model, for autistic cases. That is, they are each insufficient, on their own, for reaching the correct solution, however they are a necessary part of the configuration with other conditions, including the bar model, for reaching the correct solution.

As discussed earlier, potentially aligning with Hegarty and Mayer's (1995) conclusion that there tends to be more difficulty in constructing the visual representation – the structural phase (Ciobanu, 2015), than performing the required computations to reach the solution, the current study indicates the correct construction of the bar model an essential aspect for its use in reaching the correct solution. Although the EFs were not a condition for analysis

[20] However, as discussed earlier, reading attainment appears in some, but not all sufficient pathways, therefore can be considered as a redundant condition within the minimum formula for sufficiency.

within the present study, the findings are indicative of their potential influence in the successful utilisation of the visual representation. Consequently, it may be that the EFs form part of a necessary pathway for the successful construction and application of the bar model within mathematical word problem solving – an aspect for further research, discussed in Chapter 6.3 below.

Although the current study does not indicate the bar model alone to be either a necessary, or sufficient, condition for mathematical problem solving, reference to Thiem's (2018) conclusion, discussed above, must again be acknowledged. The current study does not therefore claim that the bar model is a causally irrelevant condition in mathematical problem solving, as it may be causally relevant under conditions beyond the scope of the current study. It does however suggest that it is not a necessary, or sufficient condition in its own right for successful mathematical problem solving for this group of pupils.

The use of QCA, as a methodological framework within the current study, is indicative of its potential to be used more widely in small-N, educational research. Despite being more widely used within larger scale studies, often within political science, public policy and comparative sociology (Rihoux, 2013; Roig-Tierno et al., 2017), the findings from the current study strengthen the argument for the wider use of this methodology within educational research. Aligning with the findings from the qualitative coding analysis and the manual analysis of configurational pathways, the data from QCA analysis within the current study, yields the same findings. Although the limitations of the findings must be acknowledged (discussed in chapter 5 and further in Chapter 6.2, below), the current study supports the use of this methodological approach for future research within the field of education, as seen through the overlap in findings represented in figure 46, above. However, the approach should be considered as appropriate only in those studies where purposive sampling is suitable and where the aim of the research is to construct empirically founded theories, emphasising causal complexity, which seek to identify those causal pathways giving rise to an outcome of interest (Hirzalla, 2020; Rihoux & Ragin, 2009; Thiem, 2018; Thomann & Maggetti, 2017).

Research question 3: Is the overall success rate in determining the correct solution to mathematical word problems greater when the bar model is employed by autistic pupils? Do autistic pupils, who have been exposed to the bar model, choose this approach when solving mathematical word problems? Do autistic pupils choose the use of visual representations when solving mathematical word problems?

The findings from the current study are consistent with those found by Bae (2013), where the number of ASD pupils selecting the use of visual representations within mathematical problem solving is low. Despite being exposed to the bar model within the mathematics lessons, for a minimum of two years, within the current study, only one pupil (ASD) chose to use the bar model when initially attempting to solve the word problem. Despite this number increasing to six, following prompting by the researcher, its use indicated limited benefit for reaching the correct solution. As discussed above, this limitation may be based on the requirement of other conditions (mathematical attainment, self-perception of mathematical ability, impaired EFs and limitations in conceptual understanding), in order for the bar model to be a useful support.

Furthermore, the findings from the current study also indicate that the use of any visual representation within mathematical problem solving is not a common choice for autistic pupils, further supporting Bae's (2013) conclusion that the use of visual representations is not associated with the problem-solving abilities of autistic (or NT) pupils.

6.2 Contribution to knowledge

The current thesis contributes academically, empirically, methodologically, and pedagogically to knowledge in the field of autism and mathematics teaching and learning.

As discussed in Chapter 1, although significant research has been previously carried out to determine the overall problem solving and mathematical ability of pupils with autism and the effectiveness of short-term interventions (Aagten-Murphy et al., 2013; Agrawal, 2013; Bae et al., 2015), there remains a distinct lack of research to 'bridge the gap between understanding the nature of academic achievement for individuals with ASD and working with educators to create practices that support autistic individuals to achieve academic success' (Keen et al., 2015, p. 17). Furthermore, in addition to the lack of research into specific pedagogical approaches to support autistic pupils within mathematics, there is a significant gap in the current body of empirical studies and research, into the effective use of the bar model, as one type of visual representation within mathematics (Aagten-Murphy et al., 2013; Agrawal, 2013; Bae, 2013; Tzanakaki et al., 2014; Wallace et al., 2019; Whitby & Mancil, 2009). Perhaps of more significance, is the current gap in understanding the specific factors, or combination of factors, directly linked with, or as predictors of, mathematical success for this group of pupils (Keen et al., 2015; Powell et al., 2019; Wei et al., 2015).

The current thesis has begun to address these gaps through the identification of the key contextual conditions and mechanisms, which underpin mathematical problem solving success and successful utilisation of the bar model within autistic pupils: mathematical attainment at ARE, or above; high levels of self-perception of mathematical ability; a limited number of EF deficits (namely attention and working memory); and a secure conceptual understanding of the mathematical concepts embedded within the word problem.

Findings from the current study identified the need to consider the factors, and combination of factors, above when supporting autistic pupils with mathematical problem solving. The presence of specific EF deficits: attention and working memory, can be seen to potentially preclude general mathematical attainment and the development of conceptual understanding, which, in turn, may impede autistic pupils' ability to benefit from the bar model, as one type of visual representation within mathematics, as well as impact upon the overall word-problem solving success rate. Furthermore, the current study has highlighted the complexity of successful mathematical problem solving for this group of pupils, through the identification of combinations of conditions, impacting upon overall problem-solving success. Consequently, in addition to contributing to the body of empirical and academic literature within this field, the current study provides implications and recommendations for mathematics practice within the classroom, as discussed further in Chapter 6.4.

Finally, through the explorative use of QCA, as a methodological framework, the current study has added to the limited empirical evidence base within this field. Significantly underused as a research methodology within small-N, educational research, the current study advocates the use of this methodological approach within future educational studies, which have their focus on comparative analysis and the identification of configurational pathways within studies drawing on complex factors and interactions of factors.

6.3 Limitations of the study

As with all empirical studies, the current research has limitations, which must be acknowledged. Although many of the limitations of the current study have been identified and referred to throughout, this section provides an overview of these for the reader.

One of the main limitations of the current study, was the sample size (N=9) used. Due to the specific selection (and deselection) criteria used, to focus the study on a manageable number of conditions for analysis, the small sample size gave rise to the issue of limited diversity within QCA analysis. This was particularly prominent within the NT subgroup, who were selected to provide a small comparison group. As only two NT participants were included, QCA analysis within this group was not possible as the number of logically possible combinations of conditions instantly exceeded the volume of empirical data available. Although limited diversity was an issue, which also presented itself within the autistic subgroup (N=7), the volume of empirical data available was sufficient to generate plausible configurational pathways for analysis. The similarities between the three solution types (discussed in Chapter 4.6) strengthened the reliability of the findings within the current study.

Related to the issue of limited diversity, was the limited number of conditions for analysis used within the current study. Although the conditions selected were drawn from the existing body of literature and empirical research, it must be acknowledged that other factors are likely to be influential on the overall problem-solving success within mathematical problem solving. Consequently, no claim that additional contextual factors and conditions, beyond the scope of the current study, is made in terms of their potential influence on mathematical problem solving for autistic or NT pupils. However, it is not possible to consider an infinite number of conditions within any study and as Thiem (2018) reminds us, the current study acknowledges that, 'just because some factor does not appear in the final model, does not mean that this is causally irrelevant to the analysed outcome; it simply means that, conditional on the available data, there exists no evidence for the causal relevance of that factor' (p.3).

A further limitation, acknowledged earlier within the thesis, was the use of only one mathematical word problem for each participant within the study. The justification for this was based on ethical grounds, to reduce the levels of stress or anxiety experienced by the participants (Leatherland, 2017), as well as the desire to collect in-depth, high quality data. The potential additional anxiety and stress placed on the participants by providing them with multiple questions to solve was likely to reduce the overall quality of the data obtained, as well as impact upon the attention and working memory of the pupils (which the current study has identified as being significant mechanisms within the mathematical problem-solving process). The limitation on the number of problems posed to the participants also restricted a detailed analysis on the consideration of the type of bar model used, and its impact upon successful mathematical problem solving. Although analysis was carried out on this aspect of the study, the data is limited as this was not used as a key condition for analysis within the current study (because of further limited diversity, discussed above). As discussed further in the following section, this is an aspect which would benefit from future research.

One of the key selection criteria used within the current study was that all participants had been exposed to the bar model within mathematics lessons for a minimum of two years. Although this criterion ensured the familiarity of the pupils with the approach, it did not (and cannot, unless carried out within a controlled experimental design) control for the manner in which, and the frequency of, the use of the bar model within mathematics teaching. Therefore, it must be acknowledged that the quality and frequency of exposure to the bar model within mathematics lessons, was most likely not equitable across cases.

Consequently, this factor may have influenced the findings from the current study, as some pupils may have developed a deeper understanding of the conceptual framework of the bar model and have had more exposure to its use within mathematical problem solving than others. However, what the current study does provide is a series of recommendations for practitioners to consider when using the bar model within their classroom practice, to maximise its potential influence on successful mathematical problem solving (discussed in Chapter 6.4, below).

To conclude, the current study has limitations, which have been acknowledged throughout, and summarised within this section. However, to maximise the validity of the study, transparency and rigour have been applied to the research, as discussed throughout, and in detail in Chapter 3.8.

6.4 Implications for practice and future research

The current study has indicated encouraging findings regarding the key contextual conditions and mechanisms underpinning successful mathematical problem solving for autistic pupils. However, as discussed in chapter 4, such findings must be acknowledged to be temporal, based on the design of the current study, which provided a 'snapshot' of pupils' performance on the mathematical word problem-solving task at a fixed point in time. Such a fixed time assumes the presence (or absence) of specific conditions and the activation of specific mechanisms at the time of data collection within the study. Nevertheless, it may be the case that when such conditions and mechanisms are considered over a period, through implementation of the recommendations made within this chapter, such changes in performance may indeed give rise to long-term, or structural changes in the brain: another potential area for future study.

The findings indicate the significance of the EFs, particularly attention and working memory, within this process. Practitioners should consider these deficits carefully within their practice and acknowledge that support with these skills, may contribute a vital aspect of enhancing the mathematical problem-solving performance of these pupils. Furthermore, a focus on these skills may be beneficial to developing pupils' conceptual understanding within mathematics and the successful application of the bar model (and potentially other types of representation), as a visual representation to support mathematical problem solving.

Although the bar model is now a common representation drawn upon within mathematics teaching and learning, practitioners should consider the importance of ensuring the secure conceptual understanding of the learners both within the structure of the word problems, and the phases of application of the bar model. The importance of developing both procedural and conceptual understanding amongst the learners has been highlighted as a significant aspect of successful utilisation of the bar model within mathematics teaching and learning.

A key condition, evident in successful mathematical problem solving, which has emerged from the current study, is that of pupils' self-perception of their own mathematical ability. Consequently, practitioners should consider this within their classroom practice and consider the role of this epistemic emotion on overall attainment and success. It may be beneficial for practitioners to consider ways in which those pupils with a low self-perception of their mathematical ability may be enhanced through classroom practice.

Throughout the thesis, suggested areas for future research have been acknowledged. However, this final section provides an overview of these aspects, which have emerged from the current study.

In terms of the potential future benefits of the bar model, it is worth considering the success of alternative universal representations, such as the empty number line, which has been used increasingly from the 1990s (Bobis & Bobis, 2005). Research into the use, and success, of the empty number line suggests a development in mathematical conceptual understanding and reasoning may be developed through the use and application of such a representation (Gravemeijer, 2020). In line with this, as the bar model establishes greater foundations, as a visual representation within mathematics' teaching and problem solving, research into the longer-term impact and implications of this representation, like those carried out with the empty number line, is recommended as an area for future research.

As discussed above, the sample size and number of word problems used within the current study have been acknowledged as limitations. The findings from the current study provide encouraging results for the teaching and learning of mathematical problem solving for autistic pupils, however, further research would benefit from repeating the study with an extended the sample size and a greater number of word problems to increase the statistical power and to check for consistency of the findings. Additionally, the use of a greater number of word problems, drawing on different versions of the bar model, would enable further insight into the type of bar model used to be explored, in terms of its potential influence of successful utilisation and application of the bar model. A larger sample of autistic cases would allow for investigation into the key conditions established within the current study to be checked for consistency within a larger population. A larger sample of NT participants would also enable a comparison to be made, through QCA analysis, to establish whether the findings within the autistic sample are similar to, or to identify any significant differences with, those found within the NT population. Furthermore, the inclusion of a sample of girls within future research would enable any differences, and influences of gender in mathematical problem solving, to be established.

In order to maintain the focus on autistic influences on mathematical problem solving, the current study excluded those pupils who displayed co-morbid conditions with autism. Further research would benefit from widening the sample to include those cases, where common co-morbid conditions, such as epilepsy and ADHD, were present. This would enable a deeper understanding of the wider contextual factors and mechanisms, which may influence the overall problem-solving ability of autistic pupils.

One of the key findings from the current study, was the significant influence in pupils' self-perception of mathematical ability, with respect to their overall problem-solving performance. As discussed in Chapter 5.1, consideration must be paid to the potential interplay of this condition with that of mathematical attainment. An area for future research would benefit from focusing on the potential relationship and interplay between these two conditions, both with autistic and NT pupils.

Finally, the current study highlighted the potential importance of the mechanisms of compensation amongst autistic individuals, with respect to masking some of the EF deficits. As this mechanism appeared to be relevant to the overall problem-solving performance of autistic pupils, further research into the overall influence of compensation as a mechanism, is recommended for further investigation from this study's findings.

Based on the recommendations, discussed above, a future area of research would benefit from a large-scale study, including a much larger and broader range of cases, in order to explore a greater number of conditions through QCA analysis. Utilisation of a larger and broader sample would enable the study of a wider range of factors, some of which have been identified in the current study, to explore causal conditions and configurations of causal conditions within mathematical word problem-solving amongst the autistic population.

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Appendices (Links to online repository)

All appendices are stored in an online repository: Figshare. Links to each of the appendices can be found with the references below. Appendices x-xxii are embargoed due to containing personal, confidential, or sensitive data sets.

Appendix i: Participant Background Profile Sheet (collected from class teacher)

Thompson, Shaun (2021): Appendix i: Participant background profile sheet. University of Leicester. Figure. https://doi.org/10.25392/leicester.data.15022803.v1

Appendix ii: Example of completed participant background profile sheet (SchCP1)

Thompson, Shaun (2021): Appendix ii: Example of completed participant background profile sheet (SchCP1). University of Leicester. Dataset. https://doi.org/10.25392/leicester.data.15022869.v1

Appendix iii: Participant (pupil) observation discussion prompt

Thompson, Shaun (2021): Appendix iii: Participant (pupil) observation discussion prompt. University of Leicester. Figure. https://doi.org/10.25392/leicester.data.15022881.v1

Appendix iv: Pupil mathematical word problem solving task (Year 3)

Thompson, Shaun (2021): Appendix iv: Pupil mathematical word problem solving task (Year 3). University of Leicester. Figure. https://doi.org/10.25392/leicester.data.15022890.v1

Appendix v: Pupil mathematical word problem solving task (Year 4)

Thompson, Shaun (2021): Appendix v: Pupil mathematical word problem solving task (Year 4). University of Leicester. Figure. https://doi.org/10.25392/leicester.data.15022914.v1

Appendix vi: Pupil mathematical word problem solving task (Year 5)

Thompson, Shaun (2021): Appendix vi: Pupil mathematical word problem solving task (Year 5). University of Leicester. Figure. https://doi.org/10.25392/leicester.data.15022923.v1

Appendix vii: Pupil mathematical word problem solving task (Year 6)

Thompson, Shaun (2021): Appendix vii: Pupil mathematical word problem solving task (Year 6). University of Leicester. Figure. https://doi.org/10.25392/leicester.data.15022932.v1

Appendix viii: Visual Representation Observation Form (VROF)

Thompson, Shaun (2021): Appendix viii: Visual Representation Observation Form (VROF). University of Leicester. Figure. https://doi.org/10.25392/leicester.data.15022947.v1

Appendix ix: Stem sentence completion task with examples of local and global responses

Thompson, Shaun (2021): Appendix ix: Stem sentence completion task with examples of local and global responses. University of Leicester. Figure. https://doi.org/10.25392/leicester.data.15022968.v1

Appendix x: Example of interview transcript (SchAP1)

Thompson, Shaun (2021): Appendix x: Interview transcript with pupil SchAP1. University of Leicester. Dataset. https://doi.org/10.25392/leicester.data.15028278.v1

Appendix xi-xv: Pupil mathematical word problem solving answer sheet (SchAP1, SchAP2, SchBP1, SchBP2, SchBP3)

Thompson, Shaun (2021): Appendices xi-xv: Pupil working out for the mathematics task. University of Leicester. Dataset. https://doi.org/10.25392/leicester.data.15028290.v1

Appendix xvi-xix: Pupil mathematical word problem solving answer sheet (SchBP4, SchCP1, SchCP2, SchGP1)

Thompson, Shaun (2021): Appendices xvi-xix: Pupil working out to mathematical problem solving tasks. University of Leicester. Dataset. https://doi.org/10.25392/leicester.data.15029025.v2

Appendix xx-xxi: Ethical approval and ethical amendment for the study

Thompson, Shaun (2021): Appendices xx-xxi: Ethical approval. University of Leicester. Dataset. https://doi.org/10.25392/leicester.data.15028302.v1

Appendix xxii: Participant information pack (consent)

Thompson, Shaun (2021): Appendix xxii: Participant information pack. University of Leicester. Dataset. https://doi.org/10.25392/leicester.data.15028305.v1